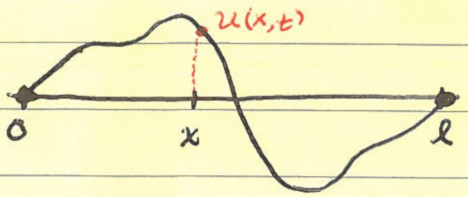
~~§2.1~~ (§2.1-2.2) — The Wave Equation on $\mathbb{R}$.

- Consider a vibrating elastic string of length $l > 0$, and spse. string is fixed at the end pts.



x = physical position on string
t = time
$u(x,t)$ = displacement of string from equilibrium at pos. x at time t.

$u(0,t) = u(l,t) = 0 \ \forall t$. (Dirichlet B.C.'s)

__Q__: If we "pluck" the string, can we model the motion?

__A__: Of course!

Here, the Kinetic energy is

Density of string (may depend on x....)

$$T = \frac{1}{2}\int_0^l \rho u_t^2 \, dx \approx \frac{1}{2} m v^2$$

and assume the potential energy is proportional to change in arclength of string, i.e. $\exists$ constant $k \in \mathbb{R} \ni$

$$\mathcal{U} = k \int_0^l \left(\sqrt{1+u_x^2} - 1\right) dx$$

Hooke's Law!!

By Hamilton's principle, i.e. princ. of ~~least~~ stationary action, the motion of the string is described by a c.p. of the "action"

$$S[u] = \int_{t_1}^{t_2} \int_0^l \underbrace{\left[\frac{1}{2}\rho u_t^2 - k\left(\sqrt{1+u_x^2} - 1\right)\right]}_{T - \mathcal{U}} dx \, dt$$

<u>Claim</u>: C.P.'s of above action satisfy a PDE!

For details, take Math 648 (Calc. of Variations). Here proceed formally: c.p.'s of S above should satisfy

describes "nearby" motion of string — If c.p., describes motion of string

$$\lim_{\varepsilon \to 0} \frac{S[u+\varepsilon v] - S[u]}{\varepsilon}$$

where v is arbitrary ftn. satisfying

(i) $v(0,t) = v(l,t) = 0 \ \forall t$ — Ends of string are fixed.

(ii) $v(x,t_1) = v(x,t_2) = 0 \ \forall x$ — "Initial" + "final" configuration of string fixed.

To calculate above, for $|\varepsilon| \ll 1$ have

$$S[u+\varepsilon v] = \int_{t_1}^{t_2}\int_0^l \Big[\frac{1}{2}\rho\underbrace{(u_t+\varepsilon v_t)^2}_{=u_t^2+2\varepsilon u_t v_t + O(\varepsilon^2)} - k\Big(\underbrace{\sqrt{1+(u_x+\varepsilon v_x)^2}}_{=\sqrt{1+u_x^2}+\varepsilon\frac{u_x v_x}{\sqrt{1+u_x^2}}+O(\varepsilon^2)} - 1\Big)\Big]\,dx\,dt$$

$$\overset{\text{Taylor expand in } \varepsilon}{=} S[u] + \varepsilon\int_{t_1}^{t_2}\int_0^l \Big[\rho u_t v_t - k\frac{u_x v_x}{\sqrt{1+u_x^2}}\Big]\,dx\,dt + O(\varepsilon^2).$$

Thus, ~~$\forall v$ satisfying above~~, u is a c.p. of S if

$$\lim_{\varepsilon\to 0}\frac{S[u+\varepsilon v]-S[u]}{\varepsilon} = \int_{t_1}^{t_2}\int_0^l \Big[\rho u_t v_t - k\frac{u_x v_x}{\sqrt{1+u_x^2}}\Big]\,dx\,dt = 0$$

for all v satisfying above conditions. Now, I.B.P. gives

$$\int_{t_1}^{t_2}\int_0^l \rho u_t v_t \,dx\,dt = \int_0^l\Big[-\int_{t_1}^{t_2}(\rho u_t)_t v\,dt + \underbrace{\cancel{v\rho u_t\Big|_{t=t_1}^{t_2}}}_{=0 \text{ by (ii)}}\Big]\,dx$$

and

$$\int_{t_1}^{t_2}\int_0^l \frac{u_x v_x}{\sqrt{1+u_x^2}}\,dx\,dt = \int_{t_1}^{t_2}\Big[-\int_0^l\Big(\frac{u_x}{\sqrt{1+u_x^2}}\Big)_x v\,dx + \underbrace{\cancel{\frac{u_x v}{\sqrt{1+u_x^2}}\Big|_{x=0}^{l}}}_{=0 \text{ by (i)}}\Big]\,dt$$

Thus, u is a c.p. of S if

$$\int_{t_1}^{t_2}\int_0^l \Big[(\rho u_t)_t - k\Big(\frac{u_x}{\sqrt{1+u_x^2}}\Big)_x\Big] v\,dx\,dt = 0$$

for every v satisfying (i) and (ii).

Only way this can happen is if

$$(\rho u_t)_t = k\left(\frac{u_x}{\sqrt{1+u_x^2}}\right)_x$$ ← Nonlinear Wave Eqn! (Allows possibility of large deviations within validity of linear Hooke's Law)

for all $x \in (0, l)$ and $t \in (t_1, t_2)$.

Note, if we assume "small deviations" of string, ie. require $|u_x| \ll 1$, then

$$\sqrt{1+u_x^2} \approx 1$$

and so equation of motion can be approx. by

$$\rho u_{tt} = k u_{xx}$$

Setting $c^2 = k/\rho$, follows u satisfies 2nd order, linear PDE

$$u_{tt} = c^2 u_{xx}.$$

This is the Wave Equation!!

☆ Note - Strauss has a very different "F=ma" derivation. I like above b/c it derives first a nonlinear wave eqn, and then get linear wave eqn. through approximations...

- Waves on a string behave very differently on different domains, ex: $\mathbb{R}$, half line, interval. To study this, start w/ easiest (although most unphysical) case of "∞-long" string...