Math 647 — HW 9 Solutions!

§ 6.1

#11) Suppose u solves the BVP

$$(*)\begin{cases} \Delta u = f & \text{in } D \subseteq \mathbb{R}^3, \\ \frac{\partial u}{\partial n} = g & \text{on } \partial D. \end{cases}$$

Integrating the PDE over D, find

$$\iiint_D \Delta u \, dx\,dy\,dz = \iiint_D f \, dx\,dy\,dz.$$

Writing $\Delta u = \nabla \cdot (\nabla u)$, it follows by the divergence theorem (or, equiv., Green's first identity) that

$$\iiint_D \Delta u \, dx\,dy\,dz = \iint_{\partial D} \nabla u \cdot \vec{n} \, dS,$$

where $\vec{n}$ is the unit outter normal vector to ∂D. Noting that $\nabla u \cdot \vec{n} = \frac{\partial u}{\partial n}$ by definition, it follows that if BVP $(*)$ has a soln., then

$$\iiint_D f \, dx\,dy\,dz = \iint_{\partial D} g \, dS,$$

as claimed.

Note that if $D \subseteq \mathbb{R}^2$, the above condition becomes

$$\iint_D f \, dx\,dy = \int_{\partial D} g \, dS$$

while if $D \subseteq \mathbb{R}^1$, say $D = (a,b)$, then above condition becomes

$$\int_D f \, dx = g(b) - g(a).$$

§6.3

#1) (a) By the maximum principle, u is either a constant in D or the maximum and min. values occur only on ∂D. Since we are given that $u = 1 + 3\sin(2\theta)$ on ∂D, it follows that

$$\max_{\vec{x} \in D} u(\vec{x}) = 4.$$

(b) By the Poisson integral formula,

$$u(\vec{0}) = \frac{1}{2\pi}\int_0^{2\pi}\left(1 + 3\sin(2\theta)\right)d\theta = 1.$$

§7.1

#5) Let u solve the BVP

$$\begin{cases} -\Delta u = 0 & \text{in } D \subseteq \mathbb{R}^3 \\ \frac{\partial u}{\partial n} = h & \text{on } \partial D \end{cases}$$

where h is some given ftn. with $\iint_{\partial D} h\, dS = 0$. Given any real valued ftn. w on $\overline{D}$, define

$$E(w) = \frac{1}{2}\iiint_D |\nabla w|^2 d\vec{x} - \iint_{\partial D} h w\, dS.$$

If w is any such ftn., set $v = u - w$ and note

$$E(w) = E(u - v)$$

$$= E(u) - \iiint_D \nabla u \cdot \nabla v\, d\vec{x} + E(v);$$

By Green's first identity,

$$\iiint_D \nabla u \cdot \nabla v \, d\vec{x} = -\iiint_D (\Delta u) v \, d\vec{x} + \iint_{\partial D} v \frac{\partial u}{\partial n} \, dS$$

$$= \iint_{\partial D} h v \, dS$$

so that

$$E(w) = E(u) + E(v) + \iint_{\partial D} h v \, dS$$

$$= E(u) + \frac{1}{2} \iiint_D |\nabla v|^2 d\vec{x}$$

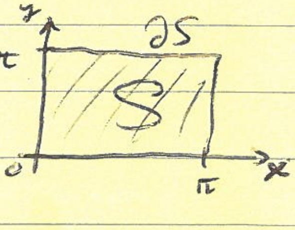
Since $\iiint_D |\nabla v|^2 d\vec{x} \geq 0$ it follows that

$$E(w) \geq E(u)$$

for all real-valued ftns. w on $\bar{D}$.

§10.1

#1) Let $S = \{(x,y) : 0 < x < \pi, \; 0 < y < \pi\}$ and consider IVBVP



$$(*) \begin{cases} u_{tt} = c^2 \Delta u & \text{in } S \text{ w/ } t > 0 \\ \partial u / \partial n = 0 & \text{on } \partial S, \; t > 0 \\ u(x,y,0) = 0 & \text{in } S \\ u_t(x,y,0) = \sin^2 x & \text{in } S \end{cases}$$

Seek separated soln. of form

$$u(x,y,t) = v(x,y) \, T(t)$$

and note v, T must satisfy

$$vT'' = c^2(\Delta v)T$$

$$\Longrightarrow \frac{T''(t)}{c^2T(t)} = \frac{\Delta v(x,y)}{v(x,y)} = -\lambda$$

for some constant $\lambda \in \mathbb{R}$. Thus, T satisfies the ODE

$$(**) \quad T'' + c^2\lambda T = 0, \quad t > 0$$

while, from given B.C.'s, v satisfies the BVP

$$(***) \quad \begin{cases} \Delta v + \lambda v = 0 & \text{in } S \\ \frac{\partial v}{\partial n} = 0 & \text{on } \partial S \end{cases}$$

Since S is a rectangle in xy-coordinates, seek non-trivial solns. of $(***)$ of form

$$v(x,y) = X(x)Y(y)$$

noting that X and Y must satisfy

$$X''Y + XY'' + \lambda XY = 0$$

$$\Longrightarrow \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = -\lambda. \quad (\star)$$

The above is only possible if there are constants $\mu, \nu \in \mathbb{R}$ such that

$$\frac{X''}{X} = -\mu, \quad \frac{Y''}{Y} = -\nu,$$

i.e. X and Y should satisfy (from given B.C.'s in $(*)$) the "separated" eigenvalue problems

$$\begin{cases} X'' + \mu X = 0, & 0 < x < \pi \\ X'(0) = X'(\pi) = 0 \end{cases}$$

and

$$\begin{cases} Y'' + \nu Y = 0, & 0 < y < \pi \\ Y'(0) = Y'(\pi) = 0 \end{cases}$$

From previous work, the eigen values for above X and Y problems are

$$\mu_n = n^2 \, , \quad \nu_m = m^2 \text{ for } n,m = 0,1,2,\ldots$$

w/ correspondings e-fns.

$$X_n(x) = \cos(nx) \, , \quad Y_m(y) = \cos(my).$$

From (☆), it follows that (∗∗∗) has a nontrivial soln. provided

$$\lambda = \lambda_{n,m} = \mu_n + \nu_m = n^2 + m^2 \, , \quad n,m = 0,1,2,\ldots$$

w/ correspondings e-fns.

$$v_{n,m}(x,y) = X_n(x)\, Y_m(y)$$
$$= \cos(nx)\cos(my) \, , \quad n,m = 0,1,2\ldots$$

With (∗∗∗) solved, return to (∗∗) to solve, for each $m,n = 0,1,2,\ldots$ the ODE

$$T_{n,m}'' + c^2 \lambda_{n,m} T_{n,m} = 0$$

$$\Longrightarrow \quad T_{n,m}(t) = \begin{cases} A_{00} + B_{0,0}\, t \, , & \text{if } n = m = 0 \\ A_{n,m}\cos(\sqrt{\lambda_{n,m}}\, ct) + B_{n,m}\sin(\sqrt{\lambda_{n,m}}\, ct), & n,m = 1,2,3\ldots \end{cases}$$

So, by linearity, any ftn. of form

$$u(x,y,t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} v_{n,m}(x,y)\, T_{n,m}(t)$$

$$= A_{0,0} + B_{0,0}\, t$$

$$+ \sum_{\substack{n,m=0 \\ n^2+m^2\neq 0}}^{\infty} \cos(nx)\cos(my)\left[A_{n,m}\cos(\sqrt{\lambda_{n,m}}\, ct) + B_{n,m}\sin(\sqrt{\lambda_{n,m}}\, ct)\right]$$

will solve the given PDE + B.C.s in (*).
To satisfy the I.C., need to choose $A_{n,m}$ and $B_{n,m}$ above so that

$$0 \overset{!}{=} A_{0,0} + \sum_{\substack{n,m=0 \\ n^2+m^2\neq 0}}^{\infty} A_{n,m}\cos(nx)\cos(my)$$

and

$$\sin^2(x) \overset{!}{=} B_{0,0} + \sum_{\substack{n,m=0 \\ n^2+m^2\neq 0}}^{\infty} B_{n,m}\cdot c\sqrt{\lambda_{n,m}}\,\cos(nx)\cos(my).$$

Since the ftns. $\{\cos(nx)\cos(my)\}_{m,n=0}^{\infty}$ are mutually orthogonal on S, it follows that

$$A_{n,m} = 0 \quad \text{for all } n,m = 0,1,2,\ldots$$

$$B_{n,m} = 0 \quad \text{if } m \neq 0 \quad \left(\text{since } \sin^2(x) \text{ is independent of variable } y\ldots\right).$$

Further, since

$$\sin^2(x) = \tfrac{1}{2}\left(1 - \cos(2x)\right)$$

follows that must choose

$$B_{0,0} = \tfrac{1}{2}, \quad B_{2,0} = \frac{-1}{2c\sqrt{\lambda_{2,0}}} = -\frac{1}{4c}$$

and

$$B_{n,0} = 0 \quad \text{for } n \neq 0, 2.$$

Thus, the soln. of the given IVBVP is

$$u(x,y,t) = \frac{t}{2} - \frac{1}{4c}\cos(2x)\sin(2ct).$$

§10.2

#4) Using polar coordinates in space, can rewrite the wave eqn. as

$$u_{tt} = c^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right), \quad c>0.$$

Seeking a soln. of form $u(r,\theta,t) = e^{-i\omega t} f(r)$, follows ω and f must be chosen so that

$$-\omega^2 f = c^2\left(f'' + \frac{1}{r}f'\right)$$

or, equivalently,

$$f'' + \frac{1}{r}f' + \frac{\omega^2}{c^2}f = 0.$$

From class, we know every bounded soln. of above ODE is of form

$$f(r) = C\, J_0\left(\frac{|\omega| r}{c}\right)$$

Follows all bounded solns. of wave eqn. of given form must be of form

$$u(r,t) = C e^{-i\omega t} J_0\left(\frac{|\omega| r}{c}\right)$$

for some constant $C \in \mathbb{R}$.