Math 647 – Applied PDEs
Homework 8 – Extra Problem

Spring 2019

Let $D = \{(x, y) : \ x^2 + y^2 < 1\} \subset \mathbb{R}^2$ and consider the Neumann BVP

$$\begin{cases} \Delta u = 0, & (x, y) \in D \\ \dfrac{\partial u}{\partial n} = h, & x^2 + y^2 = 1, \end{cases}$$

where $\frac{\partial u}{\partial n} = \nabla u \cdot n$ denotes the normal derivative of u on ∂D, n is the unit normal vector to ∂D, and where h is a given function.

(a) Write the above equation in polar coordinates (r, θ). For the boundary condition, notice that the normal derivative becomes $\frac{\partial u}{\partial r}$ in polar coordinates.

(b) Suppose that $\int_0^{2\pi} h(\theta) d\theta = 0$. Solve the BVP from part (a) using separation of variables, i.e. seek a solution of the form $u(r, \theta) = R(r)\Theta(\theta)$. In particular, show that there are infinitely many solutions in this case.

(c) Why would your solution method have failed in part (b) if we did not require $\int_0^{2\pi} h(\theta) d\theta = 0$. (*Hint: Carefully look where you used the assumption that* $\int_0^{2\pi} h(\theta) d\theta = 0$ *in part (b)...*)