Math 647 – HW 7 Solutions!

§4.1

#3) We are asked to solve the PDE + B.C.

$$(*)\quad \begin{cases} u_t = i u_{xx}, & 0<x<L,\ t>0 \\ u(0,t) = u(L,t) = 0, & t \geq 0 \end{cases}$$

Seeking separated solns. of form

$$u(x,t) = X(x)T(t)$$

we find that X and T must satisfy

$$X(x)T'(t) = i X''(x) T(t)$$

$$\Longrightarrow \quad \frac{X''(x)}{X(x)} = \frac{T'(t)}{iT(t)}$$

for all $x \in (0,L)$, $t>0$. Thus, there must be some constant $\lambda \in \mathbb{R}$ such that

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{iT(t)} = -\lambda$$

and hence X and T should satisfy the ODE's

$$(**)\quad \begin{cases} X''(x) + \lambda X(x) = 0, & 0<x<L \\ T'(t) + i\lambda T(t) = 0, & t>0. \end{cases}$$

Now, to satisfy the B.C.'s, note

$$u(0,t) = X(0)T(t) \overset{!}{=} 0 \Longrightarrow X(0) = 0$$

$$u(L,t) = X(L)T(t) \overset{!}{=} 0 \Longrightarrow X(L) = 0$$

(To avoid trivial soln!)

and hence B.C.'s on u in $(*)$ give B.C.'s on X eqn. in $(**)$:

$$(***)\quad \begin{cases} X''(x) + \lambda X(x) = 0, & 0<x<L \\ X(0) = X(L) = 0. \end{cases}$$

From previous work, know (***) has a non-trivial soln. if

$$\lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad \text{for some } n = 1, 2, 3, \ldots$$

w/ a non-trivial soln. associated to λ_n given by

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots$$

For each $n = 1, 2, 3, \ldots$ can now solve the T eqn. in (**):

$$T_n'(t) + i \lambda_n T(t) = 0$$

$$\Longrightarrow T_n(t) = A_n e^{-i\lambda_n t} = A_n e^{-i(n\pi/L)^2 t}, \quad n = 1, 2, 3, \ldots$$

for some constants A_n. It follows that there are ∞-many solns. of (*) of form

$$u_n(x,t) = X_n(x) T_n(t)$$

$$= A_n e^{-i(n\pi/L)^2 t} \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots$$

By linearity, follows any fn. of form

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-i(n\pi/L)^2 t} \sin\left(\frac{n\pi x}{L}\right)$$

will solve the given PDE + B.C.

Note: If we replace $t \longmapsto -it$, then above gives soln. to diffusion on $(0, L)$ w/ homogeneous Dirichlet B.C., In this sense, the Schrodinger eqn. can be thought of as diffusion with imaginary time...

§5.1

#8) We are asked to solve the IVBVP

$$(*)\quad \begin{cases} u_t = u_{xx}, & 0<x<1,\ t>0 \\ u(x,0)=\varphi(x), & 0\le x\le 1 \\ u(0,t)=0,\ u(1,t)=1, & t\ge 0 \end{cases}$$

where

$$\varphi(x) = \begin{cases} 5x/2, & 0<x<2/3 \\ 3-2x, & 2/3<x<1. \end{cases}$$

To handle the non-homogeneous B.C. at $x=1$, we first find the "equilibrium soln." $U(x)$, which solves the BVP

$$\begin{cases} 0 = U'', & 0<x<1 \\ U(0)=0,\ U(1)=1, \end{cases}$$

which clearly has $U(x)=x$ as the unique soln. So, if $u(x,t)$ solves $(*)$ then the ftn. $w(x,t)=u(x,t)-x$ solves the IVBVP

$$(**)\quad \begin{cases} w_t = w_{xx}, & 0<x<1,\ t>0 \\ w(x,0)=\varphi(x)-x, & 0\le x\le 1 \\ w(0,t)=0=w(1,t), & t\ge 0, \end{cases}$$

which is diffusion w/ homogeneous Dirichlet B.C.'s. From previous work, we know any ftn. of form

$$w(x,t)=\sum_{n=1}^{\infty} A_n e^{-(n\pi)^2 kt}\sin(n\pi x)$$

will solve the PDE + B.C.'s in $(**)$. To ~~determine~~ satisfy I.C., need to choose constants A_n so that

$$w(x,0)=\sum_{n=1}^{\infty} A_n \sin(n\pi x) \doteq \varphi(x)-x, \quad 0\le x\le 1 \text{ on } [0,1]$$

and so we need to determine Fourier sine series of $\varphi(x)-x$!

Notice

$$\varphi(x) - x = \begin{cases} \frac{3x}{2}, & 0 < x < 2/3 \\ 3(1-x), & 2/3 < x < 1 \end{cases}$$

and so the Fourier Sine coeff. of $\varphi(x) - x$ are

$$A_n = 2\int_0^1 (\varphi(x) - x)\sin(n\pi x)\,dx$$

$$= 2\left[\int_0^{2/3}\left(\frac{3x}{2}\right)\sin(n\pi x)\,dx + \int_{2/3}^1 3(1-x)\sin(n\pi x)\,dx\right]$$

$$= \text{Integrate by Parts!}$$

$$= \frac{9}{\pi^2 n^2}\sin\left(\frac{2n\pi}{3}\right)$$

Thus, the unique (why?) soln. to (**) is

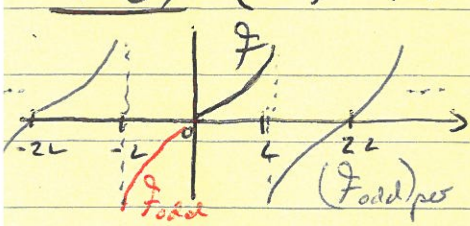
$$w(x,t) = \sum_{n=1}^{\infty}\left(\frac{9}{\pi^2 n^2}\sin\left(\frac{2n\pi}{3}\right)\right)e^{-n^2\pi^2 \cdot kt}\sin(n\pi x),$$

and hence, recalling $w = u - x$, follows the soln. to (*) is

$$u(x,t) = x + \sum_{n=1}^{\infty}\left(\frac{9}{\pi^2 n^2}\sin\left(\frac{2n\pi}{3}\right)\right)e^{-n^2\pi^2 \cdot kt}\sin(n\pi x).$$

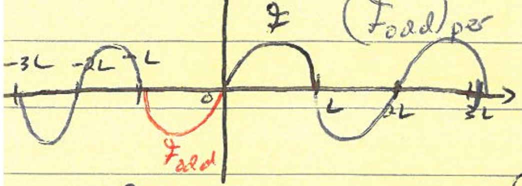
§5.4

#8) (a) To construct Fourier sine series for $f(x) = x^3$ on $[0, L]$, take $2L$-periodic extension of the odd extension f_{odd}, to get an odd $2L$-periodic ftn. $(f_{odd})_{per}$ on $\mathbb{R}$.

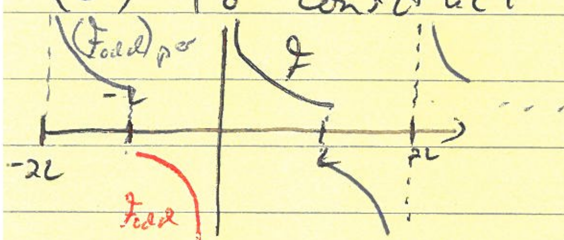


By Fourier cvgce thm., the sine series cvgs. pointwise to f on $[0, L)$. Since $(f_{odd})_{per}$ is <u>NOT</u> continuous, this convergence is not uniform. Since $\int_0^L f(x)^2\,dx < \infty$, series does converge in L^2 on $[0, L]$.

(b) To construct Fourier sine series for $f(x) = Lx - x^2$ on $[0, L]$, again take $2L$-per. extension of the odd extension f_{odd} to get a ~~ftn~~ $2L$-per. ftn. $(f_{odd})_{per}$. By Fourier convergence thm., this series converges pointwise to f on $[0, L]$. Since $(f_{odd})_{per}$ is cts. on $\mathbb{R}$ and its derivative is piecewise cts. on $\mathbb{R}$, the series converges uniformly to f on $[0, \pi]$. Finally, since $\int_0^L f(x)^2 dx < \infty$, the series cvgs. to f in L^2 on $[0, L]$.



(c) To construct Fourier sine series for $f(x) = x^{-2}$ on $[0, L]$, again take $2L$-per. extension of the odd extension (f_{odd}) to get $2L$-per. ftn. $(f_{odd})_{per}$. By Fourier convergence thm., the series cvgs. pointwise to f on $(0, L)$. Since $(f_{odd})_{per}$ is NOT continuous, the series does NOT converge uniformly on $[0, L)$. Finally, since $\int_0^L f(x)^2 dx = \infty$ the series does not converge in L^2.



Extra Problem:

(a) We seek solns. of the IVBVP

$$(*)\quad \begin{cases} u_t = u_{xx} - a(x)\,u, & 0<x<L,\ t>0 \\ u(0,t) = u(L,t) = 0, & t \geq 0 \\ u(x,0) = \varphi(x), & 0 \leq x \leq L \end{cases}$$

where $L>0$ and φ and a are given fns.
If we seek separated soln. of form

$$u(x,t) = X(x)\,T(t)$$

we see that X and T must satisfy

$$X(x)\,T'(t) = X''(x)\,T(t) - a(x)\,X(x)\,T(t)$$

$$\Longrightarrow \quad \frac{T'(t)}{T(t)} = \frac{X''(x) - a(x)\,X(x)}{X(x)}$$

for all $0<x<L$, $t>0$. Thus, there must be a constant $\lambda \in \mathbb{R}$ such that

$$\frac{T'(t)}{T(t)} = \frac{X''(x) - a(x)\,X(x)}{X(x)} = -\lambda$$

and hence X and T should satisfy the ODE's

$$(**)\quad \begin{cases} X''(x) + (\lambda - a(x))\,X(x) = 0, & 0<x<L \\ T'(t) + \lambda\,T(t) = 0, & t>0 \end{cases}$$

As usual, the B.C.'s on u in $(*)$ yield homogeneous Dirichlet B.C.'s for X in $(**)$, and hence X must satisfy BVP

$$(***)\quad \begin{cases} X''(x) + (\lambda - a(x))\,X(x) = 0, & 0<x<L \\ X(0) = X(L) = 0 \end{cases}$$

(b) Let λ be an eigenvalue for (***) w/ non-trivial eigenfunction $X(x)$. If we multiply the PDE in (***) by $\overline{X}(x)$ (the complex conjugate of $X(x)$) and integrate over $[0, L]$ we find (after rearranging)

$$-\int_0^L \overline{X}(x) X''(x)\,dx + \int_0^L a(x)\overline{X}(x)X(x)\,dx = \lambda \int_0^L \overline{X}(x)\,X(x)\,dx.$$

Using integration by parts,

$$\int_0^L \overline{X}(x) X''(x)\,dx = \overbrace{\overline{X}(x)X'(x)\Big|_{x=0}^{L}}^{\to 0 \text{ (by B.C.)}} - \int_0^L \overline{X'}(x)\,X'(x)\,dx.$$

Noting that $\overline{X'(x)}\,X'(x) = |X'(x)|^2 \geq 0$ and $\overline{X}(x)X(x) = |X(x)|^2 \geq 0$ for all x, can write above as

$$(\square)\quad \int_0^L |X'(x)|^2\,dx + \int_0^L a(x)|X(x)|^2\,dx = \lambda \int_0^L |X(x)|^2\,dx.$$

Since the Left Hand Side is real, and since $\int_0^L |X(x)|^2\,dx$ is real and not zero (else X would be ~~the~~ trivial soln), follows that it must be that

$$\lambda \in \mathbb{R}.$$

(c) By eqn. $(\square)$ above the eigenvalues must satisfy $\lambda \geq 0$ since $a(x) \geq 0$ implies

$$\lambda = \frac{\left(\int_0^L |X'|^2\,dx + \int_0^L a(x)|X|^2\,dx\right)}{\int_0^L |X|^2\,dx} \geq 0.$$

Further, $\lambda = 0$ is an e.v. if and only if ~~we~~ ~~can~~

there exists a non-trivial fcn. $\underline{X}$ on $[0,L]$ such that

$$\int_0^L |\underline{X}'(x)|^2 dx + \int_0^L a(x)|\underline{X}(x)|^2 dx = 0.$$

Since $a(x) \geq 0$, this is impossible unless $\underline{X}(x) = 0$ for all $x \in [0,L)$ and hence $\lambda = 0$ is <u>NOT</u> an eigenvalue when $a(x) \geq 0$. In particular, it follows that all the e.v.'s for $(***)$ satisfy $\lambda > 0$ when $a(x) \geq 0$.

(d) This is clear from part (c)...