Math 647 – Applied PDEs
Homework 7 – Extra Problem

Spring 2019

Extra Problem: Consider the following diffusion problem on a bounded interval:

$$\begin{cases} u_t = u_{xx} - a(x)u, & 0 < x < L, \quad t > 0 \\ u(0,t) = u(L,T) = 0 \\ u(x,0) = \phi(x) \end{cases}$$

where $L > 0$ and the functions ϕ and a are given.

(a) Show that the method of separation of variables is applicable to this problem by reducing the given PDE and boundary conditions to two ODE's coupled by a parameter λ, one of which is supplemented with a set of boundary conditions. Do NOT attempt to solve these ODE's.

(b) Prove that if $a(x)$ is a real-valued function, that is $a(x) \in \mathbb{R}$ for all $x \in [0, L]$, then all the eigenvalues of the corresponding eigenvalue problem from part (a) must be real.
Hint: Note that, to start, you don't even know if the eigenvalues are real, and hence the corresponding eigenfunctions may not be real. In general, it may be that $\lambda \in \mathbb{C}$ and that the associated eigenfunction may be complex-valued, i.e. $X(x) \in \mathbb{C}$ for all $x \in [0, L]$. For this problem, begin by multiplying the ODE for X from part (a) by[1] $\bar{X}$ and integrate the result over $[0, L]$. By integrating by parts, you should find that

$$\int_0^L |X'(x)|^2 dx + \int_0^L a(x)|X(x)|^2 dx = \lambda \int_0^L |X(x)|^2 dx$$

Taking the real and imaginary parts of the above equation, conclude that $\lambda \in \mathbb{R}$.

(c) Assuming that $a(x) \geq 0$ for all $x \in [0, L]$, prove that all the eigenvalues λ for the eigenvalue problem from part (a) must be positive.
Hint: Use your calculations from part (b) to first show if λ is an eigenvalue, then $\lambda \geq 0$. Then, show that if $\lambda = 0$ then the eigenvalue problem only has the trivial solution.

(d) Taking $a(x) = 1$ for all $x \in [0, L]$, conclude that all the eigenvalues of the given problem are strictly positive.
Remark: This follows immediately from part (c). However, this shows that there is some abstract non-sense that can sometimes be useful so you don't always have to consider $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$ separately. Furthermore, note in our class we never discussed HOW we knew that $\lambda \in \mathbb{R}$... we just assumed it and moved on. This exercise shows, in fact, that the eigenvalues λ MUST be real for this eigenvalue problem.

[1] Note if $X = U + iV$ with U and V real-valued functions, then $\bar{X} = U - iV$. In particular, $X\bar{X} = |X|^2$.