Math 647 — HW 6 Solutions!

§5.1

#4) Since $|\sin(x)|$ is even, we have for all $n = 1, 2, 3, \ldots$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin(x)| \sin(nx)\,dx = 0.$$

Similarly, we can compute

$$A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin(x)|\,dx = \frac{2}{\pi}\int_0^{\pi} \sin(x)\,dx \quad \left(\begin{array}{c}\text{Since} \\ \sin(x)>0 \text{ for} \\ x\in(0,\pi)\end{array}\right)$$

$$= \frac{2}{\pi}\left(-\cos(x)\Big|_{x=0}^{\pi}\right) = \frac{2}{\pi}\left(-(-1)+1\right) = \frac{4}{\pi}$$

and for $n = 1, 2, 3, \ldots$

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin(x)| \cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\Big(\sin((n+1)x) - \sin((n-1)x)\Big)\,dx$$

$$= \frac{1}{\pi}\left(\frac{-1}{n+1}\cos((n+1)x)\Big|_{x=0}^{\pi} + \frac{1}{n-1}\cos((n-1)x)\Big|_{x=0}^{\pi}\right)$$

$$= \frac{1}{\pi}\left(\frac{-1}{n+1}\Big(\cos((n+1)\pi)-1\Big) + \frac{1}{n-1}\Big(\cos((n-1)\pi)-1\Big)\right)$$

$$= \frac{1}{\pi}\left(\frac{-1}{n+1}\Big((-1)^{n+1}-1\Big) + \frac{1}{n-1}\Big((-1)^{n+1}-1\Big)\right)$$

$$= \frac{1}{\pi}\left(\frac{2}{n^2-1}\right)\Big((-1)^{n+1}-1\Big)$$

$$= \begin{cases} \dfrac{-4}{\pi(n^2-1)}, & n \text{ even} \\ 0, & n \text{ odd.}\end{cases}$$

Thus, the Fourier series for $|\sin(x)|$ on $[-\pi,\pi]$ is

$$\frac{2}{\pi} + \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} \frac{-4}{\pi(n^2-1)} \cos(nx)$$

or, more precisely,

$$|\sin(x)| \sim \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{-4}{\pi((2n)^2-1)} \cos(2nx)$$

valid on $[-\pi,\pi]$. Note since $|\sin(x)|$ is cts. on $\mathbb{R}$, the Fourier convergence thm. implies that

$$|\sin(x)| = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{-4}{\pi(4n^2-1)} \cos(2nx)$$

for all $x \in [0,\pi]$. In particular, at $x=0$ this gives

$$0 = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2-1}$$

$$\Longrightarrow \sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}.$$

Similarly, at $x = \pi/2$ we find

$$1 = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(n\pi)}{4n^2-1}$$

$$\Longrightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1} = \frac{2-\pi}{4}.$$

#9) From class, or eqn. (4.2.7) in Strauss, we know any ftn. of form

$$u(x,t) = \frac{A_0}{2} + \frac{B_0}{2} t + \sum_{n=1}^{\infty} \left(A_n \cos(nct) + B_n \sin(nct)\right) \cos(nx)$$

will solve the given PDE + B.C. To satisfy the I. C., need to choose constants A_n and B_n so that for $x \in [0, \pi]$ have

$$u(x,0) = 0 \stackrel{!}{=} \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nx) \qquad (i)$$

and

$$u_t(x,0) = \cos^2(x) \stackrel{!}{=} \frac{B_0}{2} + \sum_{n=1}^{\infty} (nc) B_n \cos(nx). \qquad (ii)$$

Clearly (i) is satisfied by taking $A_n = 0$ for all $n = 0, 1, 2, \ldots$ To determine constants B_n, need the Fourier Cosine series for ftn. $\cos^2(x)$. Rather than computing a bunch of integrals, notice by simple trig

$$\cos^2(x) = \frac{1}{2}\left(1 + \cos(2x)\right)$$

so that (ii) holds by taking

$$B_0 = 1, \; B_1 = 0, \; B_2 = \frac{1}{2 \cdot 2c} = \frac{1}{4c}, \; B_n = 0, \; n = 3, 4, 5, \ldots$$

Thus, the soln. to the given IVBVP is

$$u(x,t) = \frac{t}{2} + \frac{1}{4c} \sin(2ct) \cos(2x).$$

§5.2

#9) Notice that if n is odd, then

$$\sin(n(x+\pi)) = -\sin(nx) \text{ for all } x \in \mathbb{R}.$$

Thus, for $n \geq 1$ odd we have

$$\int_{-\pi}^{\pi} \varphi(x)\sin(nx)\,dx$$

$$= \int_{-\pi}^{0} \varphi(x)\sin(nx)\,dx + \int_{0}^{\pi} \varphi(x)\sin(nx)\,dx$$

$$= \int_{0}^{\pi} \varphi(x+\pi)\sin(n(x+\pi))\,dx + \int_{0}^{\pi} \varphi(x)\sin(nx)\,dx$$

$$= -\int_{0}^{\pi} \varphi(x)\sin(nx)\,dx + \int_{0}^{\pi} \varphi(x)\sin(nx)\,dx$$

$$= 0,$$

where the third equality is true since φ is π periodic and n is odd by assumption. Thus, if

$$\varphi(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$$

it follows that $a_n = 0$ for all $n \geq 1$ odd.

§5.3

#3) We seek a separated soln. of form

$$u(x,t) = X(x)\,T(t).$$

Plugging this guess into the PDE, we find X and T must satisfy

$$X(x)\,T''(t) = c^2 X''(x)\,T(t),$$

which can be rewritten as

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)}.$$

Follows there exists some $\lambda \in \mathbb{R}$ such that

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda,$$

which is equivalent to saying that X and T must satisfy the ODE's:

$$(*) \begin{cases} X''(x) + \lambda X(x) = 0, & 0 < x < L \\ T''(t) + c^2 \lambda T = 0, & t \in \mathbb{R} \end{cases}$$

for some constant $\lambda \in \mathbb{R}$.

To continue, notice the boundary conditions imply

$$u(0,t) = X(0)\, T(t) = 0 \implies X(0) = 0$$
$$u_x(L,t) = X'(L)\, T(t) = 0 \implies X'(L) = 0$$

(To avoid trivial soln!)

Thus, the given B.C.'s on u translate into "mixed" B.C.'s for the X eqn. in $(*)$.
In particular, X must satisfy the ODE-BVP

$$(**) \begin{cases} X''(x) + \lambda X(x) = 0, & 0 < x < L \\ X(0) = X'(L) = 0 \end{cases}$$

This BVP $(**)$ was solved in Exercise #1 in §4.2 in the previous HW. There, it was found that $(**)$ has a non-trivial soln. if and only if

$$\lambda = \lambda_n = \left(\left(n + \tfrac{1}{2}\right) \frac{\pi}{L} \right)^2, \quad n = 0, 1, 2, 3, \ldots$$

w/ corresponding non-trivial soln. (i.e. e-fcn.) given by

$$X_n(x) = \sin\left(\left(n + \tfrac{1}{2}\right) \frac{\pi x}{L} \right), \quad n = 0, 1, 2, \ldots$$

Now, for each $\lambda = \lambda_n$ above, we solve the T eqn. in $(*)$:

$$T_n'' + c^2 \lambda_n T_n = 0$$

$$\Longrightarrow T_n(t) = A_n \cos\left((n+\tfrac{1}{2})\frac{\pi c t}{L}\right) + B_n \sin\left((n+\tfrac{1}{2})\frac{\pi c t}{L}\right)$$

where the constants A_n, B_n are arbitrary.
This produces ∞-many solns. to the given PDE + B.C. of the form

$$u_n(x,t) = X_n(x)\, T_n(t), \quad n = 0, 1, 2, \dots$$

$$= \left(A_n \cos\left((n+\tfrac{1}{2})\frac{\pi c t}{L}\right) + B_n \sin\left((n+\tfrac{1}{2})\frac{\pi c t}{L}\right)\right) \sin\left((n+\tfrac{1}{2})\frac{\pi x}{L}\right)$$

By linearity, follows any ftn. of form

$$u(x,t) = \sum_{n=0}^{\infty} \left(A_n \cos\left((n+\tfrac{1}{2})\frac{\pi c t}{L}\right) + B_n \sin\left((n+\tfrac{1}{2})\frac{\pi c t}{L}\right)\right) \sin\left((n+\tfrac{1}{2})\frac{\pi x}{L}\right),$$

where the A_n and B_n are arbitrary constants, will solve the given PDE + B.C.

To satisfy the I.C., we need to choose the constants A_n and B_n to satisfy

(i) $u(x,0) = x \overset{!}{=} \sum_{n=0}^{\infty} A_n \sin\left((n+\tfrac{1}{2})\frac{\pi x}{L}\right), \quad 0 \le x \le L$

and

(ii) $u_t(x,0) = 0 \overset{!}{=} \sum_{n=0}^{\infty} (n+\tfrac{1}{2})\frac{\pi c}{L} B_n \sin\left((n+\tfrac{1}{2})\frac{\pi x}{L}\right), \quad 0 \le x \le L$

~~The second condition~~

To determine the constants A_n and B_n, we use the fact that $\{\sin((n+\frac{1}{2})\frac{\pi x}{L})\}_{n=0}^{\infty}$ is a mutually orthogonal set of fns. on $[0, L]$. In particular, if $n, m \in \{0,1,2,3,\ldots\}$ have

$$\int_0^L \sin\left((n+\tfrac{1}{2})\tfrac{\pi x}{L}\right)\sin\left((m+\tfrac{1}{2})\tfrac{\pi x}{L}\right)dx = \begin{cases} 0, & \text{if } n \neq m \\ \frac{\pi}{2}, & \text{if } n = m. \end{cases}$$

Follows that for (i) to hold, we need for $k = 0,1,2,\ldots$

$$\int_0^L x\sin\left((k+\tfrac{1}{2})\tfrac{\pi x}{L}\right)dx = \sum_{n=0}^{\infty} A_n \underbrace{\int_0^L \sin\left((n+\tfrac{1}{2})\tfrac{\pi x}{L}\right)\sin\left((k+\tfrac{1}{2})\tfrac{\pi x}{L}\right)dx}_{=0 \text{ unless } n=k}$$

$$= A_k \cdot \int_0^L \sin\left((k+\tfrac{1}{2})\tfrac{\pi x}{L}\right)^2 dx$$

$$= \frac{\pi}{2}\cdot A_k$$

$$\Longrightarrow A_k = \frac{2}{\pi}\int_0^L x\sin\left((k+\tfrac{1}{2})\tfrac{\pi x}{L}\right)dx, \quad k = 0,1,2,\ldots$$

Using integration by parts, we find for each $k = 0,1,2,\ldots$ we have

$$A_k = \frac{2}{\pi}\int_0^L x\,\frac{d}{dx}\left[\frac{-L}{(k+\frac{1}{2})\pi}\cos\left((k+\tfrac{1}{2})\tfrac{\pi x}{L}\right)\right]dx$$

$$= \frac{2}{\pi}\left[\frac{-xL}{(k+\frac{1}{2})\pi}\cos\left((k+\tfrac{1}{2})\tfrac{\pi x}{L}\right)\Big|_{x=0}^{L} + \frac{L}{(k+\frac{1}{2})\pi}\int_0^L \frac{d}{dx}(x)\cdot\cos\left((k+\tfrac{1}{2})\tfrac{\pi x}{L}\right)dx\right]$$

$$= \frac{2}{\pi}\left[\frac{-L}{(k+\frac{1}{2})}\overset{\to 0}{\cos\left((k+\tfrac{1}{2})\pi\right)} + \left(\frac{L}{(k+\frac{1}{2})\pi}\right)^2 \sin\left((k+\tfrac{1}{2})\tfrac{\pi x}{L}\right)\Big|_{x=0}^{L}\right]$$

$$= \frac{2}{\pi}\left(\frac{L}{(k+\frac{1}{2})\pi}\right)^2 \sin\left((k+\tfrac{1}{2})\pi\right)$$

$$= \frac{2}{\pi}\left(\frac{L}{(k+\frac{1}{2})\pi}\right)^2\cdot(-1)^k.$$

Similarly, for each $k = 0,1,2,\ldots$ we find

$$\frac{\pi}{2}(k+\tfrac{1}{2})\cdot\frac{\pi c}{L}B_k = \int_0^L 0\cdot\sin\left((k+\tfrac{1}{2})\tfrac{\pi x}{L}\right)dx = 0$$

so that $B_n = 0$ for all $n = 0, 1, 2, \ldots$
Therefore, it follows that the soln. to the given IVBVP is given by

$$u(x,t) = \sum_{n=0}^{\infty} \frac{2}{\pi}\left(\frac{L}{(n+\frac{1}{2})\pi}\right)^2 (-1)^n \cos\left((n+\tfrac{1}{2})\frac{\pi c t}{L}\right) \sin\left((n+\tfrac{1}{2})\frac{\pi x}{L}\right).$$