Math 647 – Applied PDEs
Homework 5 – Extra Problems

Extra Problem #1: Consider the wave equation on the half-line:

$$u_{tt} = c^2 u_{xx}, \quad x > 0, \quad t \in \mathbb{R}$$

with initial data $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$ valid for all $x \geq 0$. Assume there exists a number $L > 0$ such that $\phi(x) = \psi(x) = 0$ for all $x > L$.

(a) Suppose we impose the Dirichlet boundary condition $u(0,t) = 0$ for all $t \in \mathbb{R}$. Show that the waves will be at rest, and remain so, for all $0 \leq x \leq L$ after some time $T_0 > 0$. That is, show there exists a $T_0 > 0$ such that $u(x,t) = 0$ for all $x \in [0, L]$ and $t > T_0$. Calculate T_0 in terms of L.

(b) Suppose, rather, that we impose the Neumann boundary condition $u_x(0,t) = 0$ for all $t \in \mathbb{R}$. Give an example that waves might never be at rest in the interval $[0, L]$ after the time $T_0 > 0$ calculated in part (a).

Extra Problem #2: (Based on #10 in Section 2.4 or Strauss) Consider the IVP

$$\begin{cases} u_t = k u_{xx}, & -\infty < x < \infty, \quad t > 0 \\ u(0,x) = x^2, & -\infty < x < \infty \end{cases}$$

(a) Using the general solution formula from class, express the solution of this IVP as an integral. *Do not evaluate this integral!!*

(b) Observe that if $u(t,x)$ solves the above IVP, then u_{xxx} solves the heat equation with initial condition $u(0,x) = 0$ for all $-\infty < x < \infty$.

(c) Using part (b), show that the solution $u(t,x)$ to the given IVP must be of the form

$$u(t,x) = A(t)x^2 + B(t)x + C(t)$$

for some functions A, B, C. Determine specific functions A, B, C such that this provides a solution to the given IVP.

(d) By uniqueness, your answers from parts (a) and (c) must be the same. Use this observation to deduce the value of the integral

$$\int_{-\infty}^{\infty} p^2 e^{-p^2}\, dp.$$

(*Hint: Substitute* $p = (x-y)/\sqrt{4kt}$ *in the integral in (a).*)

Extra Problem #3: Consider the following boundary value problem for Laplace's equation on the box $[0, a] \times [0, b]$:

$$\begin{cases} u_{xx} + u_{yy} = 0 \quad 0 < x < a, \; 0 < y < b \\ u(0, y) = u(a, y) = 0 \\ u(x, 0) = 0, \quad u(x, b) = g(x). \end{cases}$$

Using separation of variables, show[1] this problem has solutions of the form

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right),$$

where the constants A_n can be determined from the function $g(x)$. Give an explicit formula for the constants A_n.

[1] Here you are allowed to ignore issues of convergence for these infinite series... we will devote quite a bit of time to this analysis later...