Math 647 – HW 4 Solutions!

§2.2

#5) Suppose $u(x,t)$ solves the damped wave eqn.

$$u_{tt} - c^2 u_{xx} + r u_t = 0, \quad x \in \mathbb{R}, t \in \mathbb{R}$$

where $r > 0$ is a constant. Then the energy of u at time t is given (ρ = const. density of string.) by

$$E(t) = \frac{\rho}{2} \int_{-\infty}^{\infty} \left(u_t(x,t)^2 + c^2 u_x(x,t)^2 \right) dx.$$

Differentiating with respect to t gives

$$E'(t) = \frac{\rho}{2} \int_{-\infty}^{\infty} \left(2 u_t u_{tt} + 2c^2 u_x u_{xt} \right) dx$$

$$= \rho \int_{-\infty}^{\infty} \left(u_t u_{tt} + c^2 u_x u_{xt} \right) dx.$$

Now, integration by parts gives

As usual, assume this vanishes

$$\int_{-\infty}^{\infty} u_x u_{xt}\, dx = -\int_{-\infty}^{\infty} u_{xx} u_t \, dx + \underbrace{\cancel{u_x u_t \Big|_{x=-\infty}^{\infty}}}_{\to 0}$$

so that

$$E'(t) = \rho \int_{-\infty}^{\infty} u_t \left(u_{tt} - c^2 u_{xx} \right) dx.$$

Using the PDE satisfied by u, it follows that

$$E'(t) = \rho \int_{-\infty}^{\infty} u_t \left(-r u_t \right) dx$$

$$= -r\rho \int_{-\infty}^{\infty} u_t^2 \, dx.$$

Since $r > 0$ and $\int_{-\infty}^{\infty} u_t^2 dx \geq 0$ for all $t \in \mathbb{R}$, it follows $E'(t) \leq 0$ for all $t \in \mathbb{R}$, which implies that E is a non-increasing fcn. of time.

§2.3

#1) If $u(x,t) = 1 - x^2 - 2kt$ is our soln. to the heat eqn. on

$$R = \{0 \le x \le 1,\ 0 \le t \le T\},$$

then, by the maximum principle, the max and min values of u on R must occur either when

$$t = 0 \quad \text{and} \quad 0 \le x \le 1,$$

or $$x = 0 \quad \text{and} \quad 0 \le t \le T,$$

or $$x = 1 \quad \text{and} \quad 0 \le t \le T.$$

Well, when $t = 0$ we have

$$u(x,0) = 1 - x^2$$

which has

$$\max_{x \in [0,1]} u(x,0) = u(0,0) = 1$$

and $$\min_{x \in [0,1]} u(x,0) = u(1,0) = 0.$$

Similarly, when $x = 0$ we have

$$u(0,t) = 1 - 2kt$$

which has

$$\max_{t \in [0,T]} u(0,t) = u(0,0) = 1$$

and $$\min_{t \in [0,T]} u(0,t) = 1 - 2kT = u(0,T).$$

Finally, when $x = 1$ we have

$$u(1,t) = -2kt$$

which has

$$\max_{t \in [0,T]} u(1,t) = u(1,0) = 0$$

and $$\min_{t \in [0,T]} u(1,t) = u(1,T) = -2kT.$$

Thus, by the maximum principle we have

$$\max_{(x,t)\in R} u(x,t) = u(0,0) = 1$$

The largest max found above...

and

$$\min_{(x,t)\in R} u(x,t) = u(1,T) = -2hT.$$

The smallest min found above...

#4) (a) Since $u(0,t) = u(1,t) = 0 \ \forall t > 0$, and since

$$\max_{x\in[0,1]} u(x,0) = u(\tfrac{1}{2},0) = 1,$$

(Can check using calculus!)

the strong max. princ. implies

$$0 \leq u(x,t) < 1$$

for all $t > 0$ and $0 < x < 1$.

(b) For all $x \in [0,1]$ and $t \geq 0$, set $v(x,t) = u(1-x,t)$. Since u solves the IVBVP

$$\begin{cases} u_t = k\, u_{xx}, & x \in [0,1], \ t > 0 \\ u(x,0) = 4x(1-x), & x \in [0,1] \\ u(0,t) = u(1,t) = 0, & t \geq 0 \end{cases}$$

it follows that v satisfies

$$\begin{cases} v_t = k\, v_{xx}, & x \in [0,1], \ t > 0 \\ v(x,0) = u(1-x,0) = 4(1-x)x, & x \in [0,1] \\ v(0,t) = u(1,t) = 0, & t \geq 0 \\ v(1,t) = u(0,t) = 0, & t \geq 0 \end{cases}$$

Thus, u and v satisfy the same IVP and boundary conditions. By uniqueness, we must have $u(x,t) = v(x,t) \ \forall x \in [0,1], \ t \geq 0$, as claimed.

Part (c) - Not Graded...

§2.4)

#2) The solution is

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \varphi(y)\,dy$$

$$= \frac{1}{\sqrt{4k\pi t}} \left(3\int_{-\infty}^{0} e^{-(x-y)^2/4kt} dy + \int_0^{\infty} e^{-(x-y)^2/4kt} dy \right).$$

Setting new variable $z = \frac{x-y}{\sqrt{4kt}} \Rightarrow dz = -\frac{1}{\sqrt{4kt}} dy$, have

$$u(x,t) = \frac{1}{\sqrt{\pi}} \left(3\int_{x/\sqrt{4kt}}^{\infty} e^{-z^2} dz + \int_{-\infty}^{x/\sqrt{4kt}} e^{-z^2} dz \right)$$

$$= \frac{1}{\sqrt{\pi}} \left(2\int_{x/\sqrt{4kt}}^{\infty} e^{-z^2} dz + \int_{-\infty}^{\infty} e^{-z^2} dz \right).$$

Since $\int_0^{\infty} e^{-z^2} dz = \sqrt{\pi}/2$, have

$$\int_{x/\sqrt{4kt}}^{\infty} e^{-z^2} dz = \int_0^{\infty} e^{-z^2} dz - \int_0^{x/\sqrt{4kt}} e^{-z^2} dz$$

$$= \frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2} \left(\frac{2}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-z^2} dz \right)$$

$$= \frac{\sqrt{\pi}}{2} \left(1 - \operatorname{Erf}\left(\frac{x}{\sqrt{4kt}} \right) \right),$$

where $\operatorname{Erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-x^2} dx$. Thus, using $\int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi}$ it follows that

$$u(x,t) = \frac{1}{\sqrt{\pi}} \left(\sqrt{\pi} \left(1 - \operatorname{Erf}\left(\frac{x}{\sqrt{4kt}} \right) \right) + \sqrt{\pi} \right)$$

$$= 2 - \operatorname{Erf}\left(\frac{x}{\sqrt{4kt}} \right).$$

4

#3) The soln. is

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \varphi(y)\,dy$$

$$= \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt} + 3y}\,dy.$$

Now, the exponent can be rewritten as

$$\frac{-(x-y)^2 + 12kty}{4kt} = -\frac{1}{4kt}\left(y^2 - (2x+12kt)y + x^2\right)$$

Complete the square in y... $$= -\frac{1}{4kt}\left((y-x-6kt)^2 + x^2 - (x+6kt)^2\right)$$

Simplify... $$= -\frac{1}{4kt}\left((y-x-6kt)^2 - 12xkt - 36k^2t^2\right)$$

Simplify... $$= -\frac{(y-x-6kt)^2}{4kt} + 3x + 9kt.$$

Thus, by above, the soln. can be written as

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(y-x-6kt)^2}{4kt} + 3x + 9kt}\,dy$$

$$= \left(\frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(y-x-6kt)^2}{4kt}}\,dy\right) e^{3x+9kt}$$

where the last step is justified since e^{3x+9kt} is independent of y, and hence can be pulled out of the dy integral as a constant.

Finally, defining new variable

$$z = \frac{y-x-6kt}{\sqrt{4kt}} \Rightarrow dz = \frac{dy}{\sqrt{4kt}}$$

we see $$\frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(y-x-6kt)^2}{4kt}}\,dy = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2}\,dz = 1,$$

by exercise #7 in §2.4. Therefore, the soln. is

$$u(x,t) = e^{3x+9kt}.$$