Math 647 – HW 2 Solns!

§1.1

#3) (a) 2^{nd} order, linear, inhomogeneous.
(b) 2^{nd} order, linear homogeneous.
(c) 3^{rd} order, nonlinear.
(d) 2^{nd} order, linear inhomogeneous.
(e) 2^{nd} order, linear homogeneous.
(f) 1^{st} order, nonlinear.
(g) 1^{st} order, linear homogeneous.
(h) 4^{th} order, nonlinear.

§1.2

#1) Here, the characteristic curves $(t, x(t))$ are found by solving

$$x'(t) = 3/2 \Longrightarrow x(t) = \frac{3}{2}t + C$$

where $C \in \mathbb{R}$ is arbitrary. Restricting solns. $u(t,x)$ of the PDE to these characteristic curves, we set for each $C \in \mathbb{R}$

$$v(t;C) = u(t, 3/2\, t + C)$$

and note for each fixed C the PDE restricted to the char. curves can be rewritten as

$$\frac{dv}{dt} = 0 \Longrightarrow v(t;C) = f(C)$$

for some arbitrary fn. f. Undoing above restriction, we solve $C = x - \frac{3}{2}t$ and find the general soln. of the PDE is

$$u(t,x) = f\left(x - \tfrac{3}{2}t\right)$$

where f is arbitrary. To satisfy the initial condition, need to choose fn. f so that

$$u(0,x) \overset{\text{By above gen. soln}}{=} f(x) \overset{!}{=} \sin(x).$$

Thus, choosing $f(x) = \sin(x)$ find soln. to given IVP is

$$u(t,x) = \sin\left(x - \tfrac{3}{2}t\right).$$

#7(a) Here, the charateristic curves $(x, y(x))$ satisfy

$$\frac{dy}{dx} = \frac{x}{y},$$

ie. they are of form

$$(*) \quad y^2 - x^2 = C, \quad C \in \mathbb{R}.$$

Along these char. curves, set

$$v(x;C) = u(x, y(x))$$

and note the PDE becomes

$$\frac{dv}{dx} = 0 \implies v(x;c) = f(c)$$

where f is an arbitrary fn. Solving $c = y^2 - x^2$, find gen. soln. of given PDE is

$$u(x,y) = f(y^2 - x^2), \quad f \text{ arbitrary.}$$

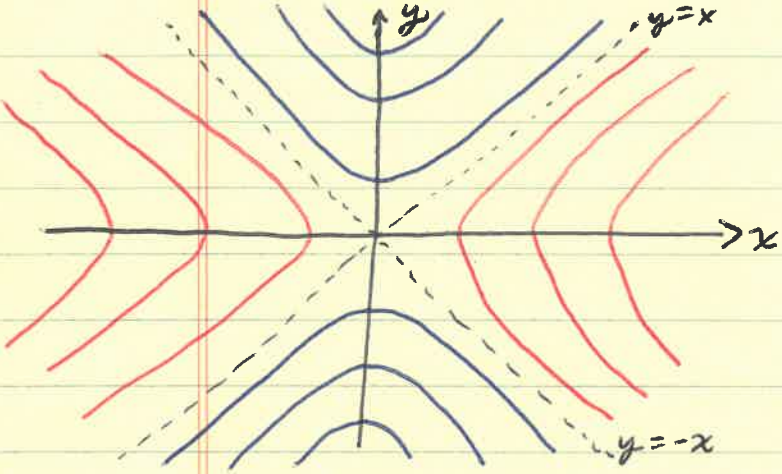
To satisfy the initial condition, need to choose fn. f so that

$$u(0,y) = f(y^2) \stackrel{!}{=} e^{-y^2}$$

$\implies$ We must choose $f(z) = e^{-z}$. Thus, soln. to given IVP is

$$u(x,y) = e^{-(y^2-x^2)}.$$

(b) To see where given I.C. uniquely defines the soln., notice the char. curves in (*) above are hyperbolas. In figure on left, curve in blue correspond to $C>0$, while those in red correspond to $C<0$.



Follows if given data defined on y-axis (as above), this data uniquely determines soln. only at pts. in xy-plane that lie on characteristics that intersect the y-axis. Therefore, above soln. is only determined (i.e. defined uniquely) in the region

$$\{(x,y)\in\mathbb{R}^2 : y^2 - x^2 \geq 0\}.$$

#8) Here, the char. curves $(x, y(x))$ satisfy

$$\frac{dy}{dx} = \frac{b}{a}$$

and hence are of the form

$$y = \frac{b}{a} x + \tilde{c}, \quad \tilde{c} \in \mathbb{R}.$$

Along these curves, set for each $\tilde{c} \in \mathbb{R}$

$$v(x; \tilde{c}) = u\left(x, \frac{b}{a} x + \tilde{c}\right)$$

and note, from PDE, v satisfies

$$\frac{dv}{dx} = -\frac{c}{a} v$$

$$\Longrightarrow \quad v(x; \tilde{c}) = f(\tilde{c})\, e^{-\frac{c}{a} x}$$

where f is arbitrary. Solving $\tilde{c} = y - \frac{b}{a} x$ it follows that the gen. soln. of the PDE is

$$u(x,y) = f\left(y - \frac{b}{a} x\right) e^{-\frac{c}{a} x},$$

where f is an arb. fn.

Note: Observing that

$$e^{-\frac{c}{a} x} = e^{\frac{-c}{a^2+b^2}(ax+by) + \frac{bc}{a^2+b^2}\left(y - \frac{b}{a} x\right)}$$

the above soln. can be written as

$$u(x,y) = \underbrace{f\left(y - \frac{b}{a} x\right) e^{\frac{bc}{a^2+b^2}\left(y - \frac{b}{a} x\right)}}_{\text{This is some fn. of } y - \frac{b}{a} x} \cdot e^{\frac{-c}{a^2+b^2}(ax+by)}$$

$$\Longrightarrow u(x,y) = g\left(y - \tfrac{b}{a}x\right) e^{-\frac{c}{a^2+b^2}(ax+by)}$$

where g is arbitry fn. This is the form of the soln. given in Strauss...

#13) Here, the char. curves $(x, y(x))$ satisfy

$$\frac{dy}{dx} = 2 \Longrightarrow y = 2x + C, \quad C \in \mathbb{R}.$$

Along these curves, we set

$$v(x; C) = u(x, 2x + C)$$

and note the PDE implies for each $C \in \mathbb{R}$ that

$$\frac{dv}{dx} + (2x - (2x+C))\, v$$

$$= 2x^2 + 3x(2x+C) - 2(2x+C)^2,$$

which simplifies to the ODE

$$\frac{dv}{dx} - Cv = -5Cx - 2C^2.$$

Multiplying above ODE by the integrating factor e^{-cx}, can rewrite above as

$$\frac{d}{dx}\left(e^{-cx} v\right) = -5cxe^{-cx} - 2c^2 e^{-cx}.$$

Integrating w.r.t. x gives

$$e^{-cx} v = 5xe^{-cx} + \frac{5}{c} e^{-cx} + 2ce^{-cx} + f(c),$$

where f is an arbitrary fn.

Solving for v and using $c = y - 2x$, follows the gen. soln. is

$$u(x,y) = 5x + \frac{5}{y-2x} + 2(y-2x) + f(y-2x)e^{(y-2x)x}$$

$$= x + 2y + \frac{5}{y-2x} + f(y-2x)e^{(y-2x)x},$$

where f is arbitrary.