Math 647 – Applied PDE
Homework 1 – Review of Prerequisites
Solutions!!

Spring 2019

1. Let $f(x, y) = \sin(xy) - x^3y + xy^4 - 12 + e^x$.

   (a) Compute f_x and f_y.
   Solution: The partial derivatives are

   $$f_x(x, y) = y\cos(xy) - 3x^2y + y^4 + e^x$$

   and

   $$f_y(x, y) = x\cos(xy) - x^3 + 4xy^3.$$

   (b) Compute[1] f_{xx}, f_{xy}, f_{yx}, and f_{yy}.
   Solution: Here, we have

   $$f_{xx}(x, y) = -y^2\sin(xy) - 6xy + e^x, \quad f_{yy}(x, y) = -x^2\sin(xy) + 12xy^2$$

   and,

   $$f_{xy}(x, y) = \cos(xy) - xy\sin(xy) - 3x^2 + 4y^3 = f_{yx}(x, y).$$

   Note the equality of mixed partials above is a consequence of Clairaut's theorem.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \cos(x) + x^2$ and let $u : \mathbb{R}^2 \to \mathbb{R}$ be given by $u(x, y) = x^2y + 2x + y^3$. Use the chain rule compute $\nabla f(u(x, y))$.
   Solution: From the chain rule, we have

   $$\partial_x f(u(x, y)) = f'(u(x, y))u_x(x, y).$$

   Since $f'(x) = -\sin(x) + 2x$ and $u_x(x, y) = 2xy + 2$, this gives

   $$\partial_x f(u(x, y)) = \left(-\sin(x^2y + 2x + y^3) + 2(x^2y + 2x + y^3)\right)(2xy + 2).$$

   Similarly, we have

   $$\begin{aligned}\partial_y f(u(x, y)) &= f'(u(x, y))u_y(x, y) \\ &= \left(-\sin(x^2y + 2x + y^3) + 2(x^2y + 2x + y^3)\right)\left(x^2 + 3y^2\right).\end{aligned}$$

   By definition then, the gradient is given by $\nabla f(u(x, y)) = \langle f_x(u(x, y)), f_y(u(x, y))\rangle$. Note, in particular, that we could write

   $$\nabla f(u(x, y)) = f'(u(x, y))\nabla u(x, y),$$

   which is simply the multi-variable chain rule.

[1] Recall that $f_{xy} = (f_x)_y$ and $f_{yx} = (f_y)_x$.

3. Find the general solution to the ODE

$$ty' + 2y = e^{t^2}.$$

Solution: Multiplying the given ODE by t, we can rewrite it as

$$\big(t^2 y(t)\big)' = te^{t^2};$$

this is really the method of "integrating factors" in action. Integrating then gives

$$t^2 y(t) = \frac{1}{2}e^{t^2} + C$$

for some constant $C \in \mathbb{R}$. Thus, the general solution is

$$y(t) = \frac{1}{2t^2}e^{t^2} + \frac{C}{t^2}, \quad C \in \mathbb{R}.$$

4. Given any $\alpha \in \mathbb{R}$, solve the initial value problem

$$y' = y^2 \cos(x), \quad y(0) = \alpha.$$

For what values of α is the solution defined for all time? (*Hint: You may need to treat* $\alpha = 0$ *and* $\alpha \neq 0$ *separately.*)
Solution: First, notice if $\alpha = 0$ then the unique solution to the given ODE is $y(t) \equiv 0$, which is clearly defined for all time. Now, if $\alpha \neq 0$ then so long as $y \neq 0$ we can rewrite the ODE as

$$\frac{dy}{y^2} = \cos(x)dx.$$

Integrating gives the solutions

$$y(t) = \frac{1}{C - \sin(x)}, \quad C \in \mathbb{R}.$$

Enforcing the condition $y(0) = \alpha \neq 0$ then gives $C = \frac{1}{\alpha}$. Thus, for all $\alpha \in \mathbb{R}$ we can write the solution of the given IVP as

$$y(t) = \frac{\alpha}{1 - \alpha \sin(x)},$$

which is well defined for all x provided the equation $1 - \alpha \sin(x) = 0$ has no solutions, i.e. if $\alpha \in (-1, 1)$.

5. Find the general solution to the differential equation

$$y'' + 2y' + 5y = 0.$$

Solution: By direct substitution, this ODE has solutions of the form $y(x) = e^{rx}$ provided the constant r satisfies the *characteristic equation*

$$r^2 + 2r + 5 = 0.$$

The roots of the characteristic equation are $r = -1 \pm 2i$. By Euler's formula

$$e^{iz} = \cos(z) + i\sin(z)$$

it follows that two (real-valued) solutions of the given ODE are given by

$$y_1(x) = e^{-x}\cos(2x), \quad y_2(x) = e^{-x}\sin(2x).$$

As these solutions are *linearly independent*, i.e. they have a non-zero Wronskian, it follows that the general solution is

$$y(x) = c_1 y_1(x) + c_2 y_2(x), \quad c_1, c_2 \in \mathbb{R}.$$

6. Consider a second order linear homogeneous ODE of the form

$$y'' + P(x)y' + Q(x)y = 0.$$

where P and Q are defined on some interval in $\mathbb{R}$. Show that the set of all solutions to this ODE forms a vector space. That is, verify each of the following:

(i) $y = 0$ is a solution.

(ii) Given any two solutions y_1 and y_2 and constants $\alpha, \beta \in \mathbb{R}$, the function $\alpha y_1 + \beta y_2$ is also a solution of the ODE.

In fact, you can show (you don't need to do this here, although it might be good to look up) that the vector space of all solutions to the above ODE has dimension 2 and a basis can be found by finding two solutions y_1 and y_2 that have a non-zero Wronskian[2] at some point where the solutions are defined.

Solution: Clearly $y = 0$ solves the given ODE. Further, if y_1 and y_2 are two solutions and $\alpha, \beta \in \mathbb{R}$ are constants, then by linearity of the derivative we have

$$\begin{aligned}
&(\alpha y_1 + \beta y_2)'' + P(x)\left(\alpha y_1 + \beta y_2\right)' + Q(x)\left(\alpha y_1 + \beta y_2\right) \\
= \quad & \alpha\left(y_1'' + P(x)y_1' + Q(x)y_1\right) + \beta\left(y_2'' + P(x)y_2 + Q(x)y_2\right) \\
= \quad & \alpha\,(0) + \beta\,(0) = 0,
\end{aligned}$$

so that $\alpha y_1 + \beta y_2$ also solves the ODE. It follows that solutions of this homogeneous ODE forms a vector space, as claimed.

[2] Recall the Wronskian is a determinant that tests for linear independence.