Linear Vs. Nonlinear PDE

Mathew A. Johnson

On the first day of Math 647, we had a conversation regarding what it means for a PDE to be linear. I attempted to explain this concept first through a hand-waving "big idea" approach. Here, we expand on that discussion and describe things precisely through the use of linear operators.

1 Operators

In our class, we will consider an *operator* as a rule that assigns to a given function a (likely different) function[1]. Here are a few examples:

(a) The identity operator, defined by $\mathcal{L}(f) = f$, i.e. $\mathcal{L}$ maps a function f to itself.

(b) The differential operator defined by $\mathcal{L}(f) = \frac{\partial f}{\partial x}$, i.e. $\mathcal{L}$ maps a function f to its partial derivative with respect to the independent variable x.

(c) The operator $\mathcal{L}(f) = af$, where a is a given real-valued function.

(d) The operator $\mathcal{L}(f) = f^2$.

(e) The operator $\mathcal{L}(f) = 2f_{yy} + 3ff_x$

Mathematically, some of the operators above are *much* better behaved than other. One of the ways that an operator can be well-behaved if by being linear.

Definition: An operator[2] $\mathcal{L}$ is a *linear operator* if it satisfies the following two properties:

(i) $\mathcal{L}(u + v) = \mathcal{L}(u) + \mathcal{L}(v)$ for all functions u and v, and

(ii) $\mathcal{L}(cu) = c\mathcal{L}(u)$ for all functions u and constants $c \in \mathbb{R}$.

If an operator is not linear, it is said to be *nonlinear*.

[1] So, operators are function-valued functions of functions...

[2] Here, I am being very sloppy with what kind of functions can be input for an operator, i.e. I am ignoring domain issues. For example, the function $f(x) = |x|$ does not lie in the domain of the operator $\mathcal{L}$ in (b) above since we can not take the derivative at $x = 0$. Here, lets just agree that if I write $\mathcal{L}(f)$ that f is assumed to be in the domain of f.

In essence, linear operators are nice because they preserve the vector space structure of their domains, i.e. if the functions belong to a vector space, then the image of the operator also forms a vector space. For us, the main distinction is that the theory of linear PDE is MUCH better developed than that for nonlinear PDE[3].

In practice, checking whether a given operator is linear or not is easy: start by checking property (ii) above (this is usually easy to check in your head), and then, if (ii) holds, check if (i) also holds. Lets look at the examples listed above.

(a) The identity operator is a linear operator since, by definition,

$$\mathcal{L}(u+v) = u + v = \mathcal{L}(u) + \mathcal{L}(v)$$

for all functions u and v. Further, given any function f and constant $c \in \mathbb{R}$ we have

$$\mathcal{L}(cf) = cf = c\mathcal{L}(f).$$

Thus, the identity operator is a linear operator.

(b) Since derivatives satisfy $\partial_x (f+g) = f_x + g_x$ and $(cf)_x = cf_x$ for all functions f, g and constants $c \in \mathbb{R}$, it follows the differential operator $\mathcal{L}(f) = f_x$ is a linear operator.

(c) This operator can be shown to be linear using the above ideas (do this yourself!!!).

(d) This operator is quickly seen to be nonlinear by noticing that property (ii) above fails. Indeed, given any function f and constant $c \in \mathbb{R}$ we have

$$\mathcal{L}(cf) = (cf)^2 = c^2 f^2,$$

which agrees with the quantity $c\mathcal{L}(f) = cf^2$ if and only if $c = 1$. Since property (ii) is supposed to hold for *every* constant $c \in \mathbb{R}$, it follows that $\mathcal{L}$ is *not* a linear operator.

(e) Again, this operator is quickly seen to be nonlinear by noting that

$$\mathcal{L}(cf) = 2cf_{yy} + 3c^2 f f_x,$$

which, for example, is not equal to $c\mathcal{L}(f)$ if, say, $c = 2$. Thus, this operator is nonlinear. Notice in this example that $\mathcal{L}$ is the sum of the linear operator $\mathcal{L}_1 = 2\partial_y^2$ and the *nonlinear* operator $\mathcal{L}_2(f) = 3ff_x$.

[3]In fact, nonlinear PDE is still an active area of current research!

2 Linear Vs. Nonlinear PDE

Now that we (hopefully) have a better feeling for what a linear operator is, we can properly define what it means for a PDE to be linear. First, notice that any PDE (with unknown function u, say) can be written as

$$\mathcal{L}(u) = f.$$

Indeed, just group together all the terms involving u and call them collectively $\mathcal{L}(u)$, and let f denote all the terms in the PDE that don't explicitly involve the function u.

Definition: The PDE $\mathcal{L}(u) = f$ is a *linear PDE* if and only if the operator $\mathcal{L}$ is a linear operator.

As an example using this definition, consider the following two PDE's:

$$u_t + u_{xxx} + uu_x = 0$$

and

$$u_t = u_{xx} + \cos(xy)u + xy^2.$$

In the first case, we can write the PDE in "operator form" as $\mathcal{L}(u) = 0$ where

$$\mathcal{L}(u) = u_t + u_{xxx} + uu_x.$$

This operator is quickly seen to be nonlinear (due to the uu_x term) since it fails property (ii) above. Thus, the first PDE listed above is nonlinear. On the other hand, the second PDE can be written in operator form as $\mathcal{L}(u) = f$ where

$$\mathcal{L}(u) = u_t - u_{xx} - \cos(xy)u$$

and $f(x, y) = xy^2$. Since $\mathcal{L}$ is a linear operator (CHECK THIS!!), the second PDE above is a linear PDE.

Further examples and discussion can be found in the course textbook, but I sincerely hope this helps clarify the notion of linearity in the context of PDE.