1st Order Linear PDE: An Example

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The "method of characteristics" attempts to solve 1st order linear PDE by trying to find curves in the domain along which the PDE reduces to an ODE. To illustrate this method, lets consider trying to solve the PDE

$$x^2 u_x + y^3 u_y = 0, \quad x \neq 0 \tag{0.1}$$

subject to the condition $u(x,1) = x^2$. To find an explicit form for the solution, suppose $u(x,y)$ solves the given PDE and let $(x, y(x))$ be a curve in the xy-plane. Restricting the solution to the curve $(x, y(x))$ we find by the chain rule that

$$\frac{d}{dx} u(x, y(x)) = u_x + y' u_y.$$

Since the function u is supposed to solve the PDE, it follows that if we now choose the curve $(x, y(x))$ to satisfy

$$y' = \frac{y^3}{x^2}, \tag{0.2}$$

then along such a curve the solution will satisfy the ODE

$$\frac{d}{dx} u(x, y(x)) = 0. \tag{0.3}$$

Now, the ODE (0.2) is separable and can be solved implicitly to give

$$\frac{1}{x} - \frac{1}{2y^2} = C, \tag{0.4}$$

where $C \in \mathbb{R}$ is an arbitrary constant of integration. By (0.3), it follows that u is constant along these characteristic curves defined by (0.4). In particular, given any $(x_0, y_0) \in \mathbb{R}^2$ the value of $u(x_0, y_0)$ is determined only by the value of $C = \frac{1}{x_0} - \frac{1}{2y_0^2}$. It follows then that the general solution of the given PDE is

$$u(x,y) = f\left(\frac{1}{x} - \frac{1}{2y_0^2}\right) \tag{0.5}$$

where f is an arbitrary function.

With the general solution in hand, we now seek the form of f to ensure that the solution satisfy the condition $u(x,1) = x^2$. To this end, evaluating (0.5) along the curve $(x, 1)$, we see that we need to require that f satisfy the condition

$$f\left(\frac{1}{x} - \frac{1}{2}\right) = x^2.$$

Setting $w = \frac{1}{x} - \frac{1}{2}$, so that $x = \frac{2}{2w+1}$, it follows that f must satisfy

$$f(w) = \left(\frac{2}{2w+1}\right)^2.$$

The above equation uniquely determines the function f, and hence the solution to the given problem is given explicitly by

$$u(x, y) = \left(\frac{2}{2\left(\frac{1}{x} - \frac{1}{2y^2}\right) + 1}\right)^2.$$

Remark: More generally, the characteristic curves for the PDE

$$u_x + h(x, y)u_y = 0$$

are found by solving the ODE

$$\frac{dy}{dx} = h(x, y).$$

Assuming this ODE can be solved, then along the characteristic curves $(x, y(x))$ the solution u of the PDE satisfies the ODE

$$\frac{d}{dx}u(x, y(x)) = 0.$$

Thus, even in this more general solution, we see that the solution u will be constant along the characteristic curves.