SIAM J. MATH. ANAL.
Vol. 42, No. 6, pp. 2681–2702

TRANSVERSE INSTABILITY OF PERIODIC TRAVELING WAVES IN THE GENERALIZED KADOMTSEV–PETVIASHVILI EQUATION*

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Abstract. In this paper, we investigate the spectral instability of periodic traveling wave solutions of the generalized Korteweg–de Vries equation to long wavelength transverse perturbations in the generalized Kadomtsev–Petviashvili equation. By analyzing high and low frequency limits of the appropriate periodic Evans function, we derive an orientation index which yields sufficient conditions for such an instability to occur. This index is geometric in nature and applies to arbitrary periodic traveling waves with minor smoothness and convexity assumptions on the nonlinearity. Using the integrable structure of the ordinary differential equation governing the traveling wave profiles, we are then able to calculate the resulting orientation index for the elliptic function solutions of the Korteweg–de Vries and modified Korteweg–de Vries equations.

Key words. transverse instability, generalized Korteweg–de Vries equation, periodic waves, generalized Kadomtsev–Petviashvili equation

AMS subject classifications. 35B35, 35B10, 35Q53

DOI. 10.1137/090770758

1. Introduction.

The Korteweg–de Vries equation

$$u_t = u_{xxx} + uu_x \tag{1.1}$$

often arises as a model for one-dimensional long wavelength surface waves propagating in weakly nonlinear dispersive media as well as the evolution of weakly nonlinear ion acoustic waves in plasmas [TRR]. When the assumption that the wave is purely one-dimensional is relaxed to allow for weak dependence in a transverse direction, one is led to a variety of multidimensional generalizations of the KdV equation. One of the most well studied weakly two-dimensional variations of the KdV equation is the Kadomtsev–Petviashvili (KP) equation [KP] given by

$$(u_t - u_{xxx} - uu_x)_x + \sigma u_{yy} = 0, \tag{1.2}$$

where the constant $\sigma = \pm 1$ differentiates between equations with positive ($\sigma = +1$) and negative ($\sigma = -1$) dispersion and the choice of σ depends on the exact physical phenomenon being described.[1] For instance, if $\sigma = +1$ (1.2) is referred to as the KP-I equation, which can be used to model waves in thin films with high surface tension, while (1.2) is called the KP-II equation in when $\sigma = -1$, which can be used to model water waves with small surface tension. Various other physical applications utilize equations of the form (1.2), such as the modeling of small amplitude internal waves and in the study of unmagnetized dusty plasmas with variable dust charge [PJ].

*Received by the editors September 10, 2009; accepted for publication (in revised form) August 18, 2010; published electronically October 21, 2010.
http://www.siam.org/journals/sima/42-6/77075.html
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The first author was partially supported by an NSF Postdoctoral Fellowship under NSF grant DMS-0902192. The second author was partially supported under NSF grants DMS-0300487 and DMS-0801745.
[1]Notice that one can always rescale the y variable to force σ to be any (nonzero) real number. However, for convenience, we will always assume that $\sigma = \pm 1$.

In many applications, appropriate scaling in the physical parameters introduces a parameter $\alpha > 0$ in the nonlinearity yielding a governing equation of the form

$$u_t = u_{xxx} + \alpha u u_x.$$

In neighborhoods of parameter space where $\alpha = 0$ one is forced to consider higher order expansions in the nonlinearity, the most natural being of the form

$$u_t = u_{xxx} + \beta u^2 u_x,$$

where $\beta \neq 0$. The choice of the sign of β must be made depending on the particular physical situation being studied. In particular, the choice of β clearly determines the structure of the stationary homoclinic/heteroclinic solutions, and hence the case of $\beta > 0$ and $\beta < 0$ define quite distinct dynamics. In the case of the modified KP (mKP) equation

$$\left(u_t - u_{xxx} - \beta u^2 u_x\right)_x + \sigma u_{yy} = 0$$

arises naturally as a weakly two-dimensional generalization. In the context of dusty plasmas with variable dust charge, the mKP equation can be derived near the "critical" density case (see [PJ] for details).

As our theory will not depend on the explicit form of the nonlinearity and to encompass as many physical applications as possible, we will most often work with the generalized KdV (gKdV) equation

$$u_t = u_{xxx} + f(u)_x \tag{1.3}$$

and its weakly two-dimensional variation of the generalized KP (gKP) equation

$$(u_t - u_{xxx} - f(u)_x)_x + \sigma u_{yy} = 0 \tag{1.4}$$

where the nonlinearity f is sufficiently smooth and satisfies general convexity assumptions. For such nonlinearities, the gKdV equation admits asymptotically constant traveling solutions, known as solitary waves, as well as traveling waves which are spatially periodic. It is the latter case we consider here. As a solution, $u(x,t)$ of (1.3) is clearly a y-independent solution of (1.4); it seems natural to question the stability of such a solution to perturbations which have a nontrivial dependence on the transverse (y) direction. Such a transverse instability analysis is the subject of the current paper: in particular, we study the spectral stability of a stable y-independent spatially periodic traveling wave solution of the gKdV equation to perturbations which are coperiodic[2] in x with low frequency oscillations in the transverse direction. To this end, we will loosely follow the general Evans function approach of Gardner [G1] with additional aspects of analysis from the more recent work of Johnson [J2].

The transverse instability of solitary waves of the KdV in the KP equation was first conducted by Kadomtsev and Petviashvili [KP], where it was found that such solutions are stable to transverse perturbations in the case of negative dispersion, while they are unstable to long wavelength transverse perturbations in the case of positive dispersion (even though they are stable in the corresponding one-dimensional problem). Moreover, in [APS] it was shown that a short wavelength cutoff for instability exists for the positive dispersion case and the dominate mode of instability was

[2]Notice by Floquet theory, one-dimensional spectral instability to coperiodic perturbations implies spectral instability to localized perturbations.

identified. Other authors have devised various techniques to demonstrate the transverse instability of KdV solitary waves in the KP-I equation: for example, see the pioneering work of Zakharov [Za], where the author utilizes the integrability of the KP-I equation via the inverse scattering transform, and the recent work of Rousset and Tzvetkov [RT2], where the authors use general PDE techniques.[3] The transverse instability of a one-dimensional stable gKdV solitary wave in the corresponding gKP equation has recently been considered in [KTN] using perturbation analysis similar to that of Kadomtsev and Petviashvili. In particular, multiple scale analysis was used to derive an evolution equation for the wave velocity to describe the slow-time response of the solitary wave in response to the long wavelength transverse perturbations. The authors conclude that for positive dispersion the solitary waves of the gKdV equation are always unstable to long wavelength transverse perturbations in the gKP equation. Moreover, it was found that for some nonlinearities the solitary waves may in fact be unstable in the case of negative dispersion.

In the case where the background solution of the gKdV is spatially periodic, there do not seem to be any results concerning the transverse instability in the gKP equation, and, moreover, the stability of such solutions in general is much less understood. This reflects the fact that the spectrum of the corresponding linearized operators is purely continuous, and hence it seems more difficult for nonlinear periodic waves of the gKdV to be stable than their solitary wave counterparts. Moreover, the periodic waves of the gKdV in general have a much more rich structure than the solitary waves: even in the case of power-law nonlinearities, it is not possible to write down a general elementary representative for all periodic solutions of the gKdV, which stands in contrast to the solitary wave theory. Nevertheless, there has been much study recently into the one-dimensional stability of such solutions under the gKdV flow with results ranging from spectral stability to localized perturbations (see [BD], [BrJ], and [HK]) to nonlinear (orbital) stability to coperiodic perturbations (see [BrJK], [DK], and [J1]). We also want to stress the fact that the periodic solutions are also physically relevant as they are used to model nonlinear wave trains: such patterns are prevalent in a variety of applications and their instabilities have been studied extensively in the literature (see, for example, the classic works of Benjamin [Be1] and [Be2], Benjamin and Feir [BF], Lighthill [L], and Whitham [W]).

The goal of this paper is to perform a transverse instability study in the case of spatially periodic traveling wave solutions which are stable to perturbations in the x-direction of unidirectional propagation. To this end, we will develop a somewhat nonstandard orientation index which detects instability of a T-periodic traveling wave of the gKdV to perturbations in the gKP equation which are T-periodic in the x-direction with low frequency oscillations in the transverse spatial variable y. Throughout this paper, we will refer to such perturbations as long wavelength transverse perturbations. This is accomplished by studying the behavior of the corresponding periodic Evans function in both the high (real spectral) frequency and low (transverse) frequency limits when the wave number of the transverse perturbation is small but nonzero. As we will see, the high frequency analysis is somewhat delicate due to a degeneracy in the KP equation. In particular, it is seen that one must take into account not only the higher order effects of the dependence of the limiting asymptotic ordinary differential equation on the background solution as in [MaZ3] and [PZ], but also the

[3]The methods of Rousset and Tzvetkov provide nonlinear transverse instability of the KdV solitary wave and have the advantage over that of Zakharov of generalizing to the full water wave problem in the presence of surface tension.

inherent averaging/cancellation effects due to the periodicity. Such a result seems new to the literature and relies on the use of block-triangularizing transformations as used in [HLZ]. Moreover, in the appendix we give a slight simplification of the of the tracking lemmas utilized in [MaZ3], [PZ], and [HLZ] by formulating the result in terms of conjugating transformations rather than invariant graphs. This viewpoint seems more closely related to the style and needs of current research (see, for example, the related treatments in [Z6] and [NZ]). As a result, we will find that the limiting behavior of the *sign* of the periodic Evans function for large (real) spectral frequency is governed precisely by the sign of the dispersion parameter σ. Thus, the stability of our periodic traveling waves depends directly on the case of dispersion chosen in the gKP equation and hence on the exact physical phenomenon being described.

The low frequency analysis utilizes matrix perturbation theory and the methods of [BrJ] and [J2]. In particular, we will explicitly construct four linearly independent stationary solutions of the linearization of the governing PDE: three of these will be given to us by variations in the traveling wave parameters, which we will use to parameterize the traveling wave solutions of (1.3), while the fourth can be constructed using standard techniques. Using variation of parameters then, we can compute the leading order variation of these four functions in the transverse wave number k for $|k| \ll 1$ and hence can determine the leading order variation of the periodic Evans function in the transverse wave number at the origin in the spectral plane. This leading order variation is expressed as a Jacobian determinant relating to the ability to parameterize nearby periodic waves of fixed wave speeds by the conserved quantities of the gKdV flow. It follows then that if this determinant has the opposite sign of that of the high frequency limit (the parameter σ), we immediately have a spectral instability to long wavelength transverse perturbations in the gKP equation. Notice this approach is somewhat different than that of [J2], where transverse instability in the generalized Zakharov–Kuznetsov equation was established by finding sufficient conditions for eigenvalues bifurcating from the origin (in k) to enter the unstable half-plane. In that case, the low frequency analysis involved considering variations of elements of the kernel of the linearized operator in both the spectral variable and in the transverse wave number. As a result, the results in [J2] concern spectral instability to long wavelength perturbations in *both* the direction of propagation of the background solution and the transverse direction. The orientation index derived in our case thus seems to be nonstandard, in the sense that it detects instabilities to perturbations which have bounded period in the x-direction and admit slow modulations in the transverse direction.

The outline of this paper is as follows. In section 2, we review the basic properties of the periodic traveling wave solutions of the gKdV equation (1.3). In particular, we will discuss a parametrization of the traveling wave solutions of (1.3), which will be useful throughout our analysis. In section 3, we conduct our transverse instability analysis by first conducting a high frequency analysis of the associated periodic Evans function and then conducting the corresponding low frequency analysis. As a result, we will have an instability index which guarantees that a periodic traveling wave solution of the gKdV is spectrally unstable to long wavelength transverse perturbations in the gKP equation. This index can be calculated exactly in several model cases, and the relevant results will be given. In section 4, we end with some closing remarks, and in the appendix we present a proof of the tracking lemma utilized in the high frequency analysis and also outline the proof of a formula crucial to the low frequency analysis.

2. Properties of periodic traveling GKdV waves. In this section, we review some of the basic properties of the periodic traveling wave solutions of the gKdV equation. For more details, see [BrJ] or [J1]. For each $c > 0$, a traveling wave with speed c is a solution of the ordinary differential equation

$$u_{xxx} + f(u)_x - cu_x = 0; \tag{2.1}$$

i.e., they are stationary solutions of (1.3) in the moving coordinate frame defined by $x+ct$. This equation is clearly Hamiltonian, and hence we can reduce it to quadrature. Indeed, by integrating (2.1) twice we see that a traveling wave profile of the gKdV must satisfy the nonlinear oscillator equation

$$\frac{u_x^2}{2} = E + au + \frac{c}{2}u^2 - F(u), \tag{2.2}$$

where F is an antiderivative of the nonlinearity f satisfying $F(0) = 0$ and a and E are constants of integration. Thus, the traveling waves form a four parameter family of solutions of (1.3) described by the constants a, E, and c together with a fourth constant of integration corresponding to a translation mode: this translation direction is simply inherited from the translation invariance of (1.3) and hence can be modded out. It follows that on open subsets of $\mathbb{R}^3 = (a, E, c)$, (2.2) admits a periodic orbit. Moreover, the boundary of these open subsets corresponds to solutions which decay asymptotically at infinity and hence can be identified with the solitary wave solutions. In particular, notice that in order for (2.2) to admit a solitary wave solution, the constant E must be fixed by the prescribed boundary conditions and we must have $a = 0$. It follows that the solitary waves form a codimension two subset of the family of traveling waves.

In general, (1.3) admits three conserved quantities. In order to define these, let

$$V(u; a, c) = F(u) - au - \frac{c}{2}u^2$$

be the effective potential arising in the nonlinear oscillator equation (2.2). Throughout this paper, we will assume the roots $u_\pm$ of the equation $E = V(u; a, c)$ are simple, satisfy $u_- < u_+$, and that $V(u; a, c) < E$ for $u \in (u_-, u_+)$. As a consequence, $u_\pm$ are C^1 functions of the traveling wave parameters a, E, and c, and, without loss of generality, we can set $u(0) = u_-$. It follows that we can express the period of a periodic solution of (2.1) via the formula

$$T = T(a, E, c) = \sqrt{2}\int_{u_-}^{u_+} \frac{du}{\sqrt{E - V(u; a, c)}}.$$

By a standard procedure, the above integral can be regularized at the square root branch points and hence represents a C^1 function of a, E, and c. Similarly, the conserved quantities of the gKdV flow can be represented as

$$\begin{aligned}
M(a, E, c) &= \int_0^T u(x)\,dx = 2\int_{u_-}^{u_+} \frac{u\,du}{\sqrt{2\,(E - V(u; a, c))}},\\
P(a, E, c) &= \int_0^T u^2(x)\,dx = 2\int_{u_-}^{u_+} \frac{u^2\,du}{\sqrt{2\,(E - V(u; a, c))}},\\
H(a, E, c) &= \int_0^T \left(\frac{u_x^2}{2} - F(u)\right)dx = 2\int_{u_-}^{u_+} \frac{E - V(u; a, c) - F(u)}{\sqrt{2\,(E - V(u; a, c))}}\,du,
\end{aligned}$$

representing the mass, momentum, and Hamiltonian, respectively. As above, these integrals can be regularized at the branch points and hence represent C^1 functions of the traveling wave parameters. As we will see, the gradients of the period and mass of the solution u will play a very large role in this paper. However, as pointed out in [BrJ], when E is nonzero gradients in the period can be interchanged for gradients of the conserved quantities via the relation

$$E\nabla_{a,E,c}T + a\nabla_{a,E,c}M + \frac{c}{2}\nabla_{a,E,c}P + \nabla_{a,E,c}H = 0,$$

where $\nabla_{a,E,c} = \langle \partial_a, \partial_E, \partial_c \rangle$. Thus, all gradients involved in the results of this paper can be expressed *completely* in terms of the gradients of the conserved quantities of the gKdV flow, which seems to be desired from a physical point of view.

We now discuss our parametrization of the family of periodic traveling wave solutions of (1.3) more carefully. A major technical assumption throughout this paper is that the period and mass provide good local coordinates for the periodic traveling waves of fixed wave speed $c > 0$. More precisely, given a periodic traveling wave $u(\cdot; a_0, E_0, c_0)$ of (1.3) with $c_0 > 0$ we assume the map

$$(a, E) \mapsto (T(a, E, c_0), M(a, E, c_0))$$

has a unique C^1 inverse in a neighborhood of $(a_0, E_0) \in \mathbb{R}^2$, which is clearly equivalent with the nonvanishing of the Jacobian determinant

$$\{T, M\}_{a,E} := \det\left(\frac{\partial(T, M)}{\partial(a, E)}\right)$$

at the point (a_0, E_0, c_0). As we will see, the sign of this Jacobian controls the low frequency analysis presented in this paper. It is worth mentioning that while this may seem like a rather obscure requirement, this Jacobian has already shown to be important in the stability theory of periodic traveling waves of the gKdV; see, for example, [BrJ], [BrJK], [J1], and [J2]. In particular, this Jacobian been computed in [BrJK] for several power-law nonlinearities and, in these cases, has been shown to be generically nonzero. Moreover, such a nondegeneracy condition should not be surprising; a similar nondegeneracy condition must often be enforced in the stability theory for solitary waves (see [Bo], [Be2], and [PW]).

3. Transverse instability analysis. We now begin our stability analysis. Let $u = u(\cdot; a, E, c)$ be a $T = T(a, E, c)$-periodic traveling wave solution of (1.3). Moreover, we assume that u is a stable solution of the one-dimensional gKdV equation.[4] As noted in the introduction, it is clear then that u is a y-independent solution of the generalized KP equation (1.4) for either $\sigma = \pm 1$. We are interested in the spectral stability of u as a solution of (1.4) to small perturbations. To this end, consider a small perturbation of u of the form

$$\psi(x, y, t) = u(x) + \varepsilon v(x, y, t) + \mathcal{O}(\varepsilon^2), \quad |\varepsilon| \ll 1,$$

where $v(\cdot, y, t) \in L^2(\mathbb{R})$ for each $(y, t) \in \mathbb{R}^2$ and $v(x, \cdot, t) \in L^\infty(\mathbb{R})$ for each $(x, t) \in \mathbb{R}^2$. Forcing ψ to solve the traveling gKP equation

$$(u_t - u_{xxx} - f(u)_x + cu_x)_x + \sigma u_{yy} = 0 \tag{3.1}$$

[4]Else the issue of transverse instability is of no interest, as instabilities to unidirectional perturbations will prevent stability to higher-dimensional perturbations.

yields a hierarchy of consistency conditions. The $\mathcal{O}(\varepsilon^0)$ equation clearly holds since u solves (3.1), and the $\mathcal{O}(\varepsilon^1)$ equation reads as

$$\partial_x \left(\partial_t + \partial_x \mathcal{L}[u]\right) v + \sigma v_{yy} = 0,$$

where $\mathcal{L}[u] = -\partial_x^2 - f'(u) + c$ is a periodic Hill operator. As this linearized equation is autonomous in both time and the spatial variable y, we may seek separated solutions of the form

$$v(x,y,t) = e^{-\mu t + iky} v(x),$$

where $\mu \in \mathbb{C}$, $k \in \mathbb{R}$, and $v \in L^2(\mathbb{R})$. This leads one to the (generalized) spectral problem

$$\left(\partial_x^2 \mathcal{L}[u] - \sigma k^2\right) v = \mu \partial_x v \tag{3.2}$$

considered on the real Hilbert space $L^2(\mathbb{R})$. We refer to the background solution u as being spectrally stable in $L^2(\mathbb{R})$ if (3.2) has no $L^2(\mathbb{R})$ spectrum with[5] $\Re(\mu) \neq 0$ for any $k \in \mathbb{R}$.

Since the coefficients of the differential operator $\mathcal{L}[u]$ are T-periodic, as they depend on the background solution u, standard results in Floquet theory imply the L^2 spectrum of (3.2) is purely continuous and consists entirely of $L^\infty(\mathbb{R})$ eigenvalues. Indeed, the fact that (3.2) can have no $L^2(\mathbb{R})$ eigenvalues is clear: writing (3.2) as a first order system of the form

$$\mathbf{Y}_x = \mathbf{H}(x;\mu,k)\mathbf{Y}$$

and letting $\Phi(x;\mu,k)$ be a matrix solution satisfying the initial condition $\Phi(0;\mu,k) = \mathbf{I}$ for all $(\mu,k) \in \mathbb{C} \times \mathbb{R}$, we define the monodromy operator, or the period map, to be

$$\mathbb{M}(\mu) := \Phi(T;\mu,k).$$

Notice that given any vector solution $\mathbf{Y}$ of (3.2), the monodromy operator is a matrix such that

$$\mathbb{M}(\mu)\mathbf{Y}(x;\mu,k) = \mathbf{Y}(x+T;\mu,k)$$

for all $(x,\mu,k) \in \mathbb{R} \times \mathbb{C} \times \mathbb{R}$. Assuming now for simplicity that $\mathbf{Y}$ is an eigenvector of $\mathbb{M}(\mu)$ with eigenvalue λ, we clearly have that

$$\mathbf{Y}(NT;\mu,k) = \mathbb{M}(\mu)^N \mathbf{Y}(0;\mu,k) = \lambda^N \mathbf{Y}(0,\mu,k).$$

Thus, if $\mathbf{Y}(x;\mu,k)$ decays as $x \to \infty$, it must become unbounded as $x \to -\infty$. Thus the best we can hope for is for $\mathbf{Y}(x;\mu,k)$ to remain bounded on $\mathbb{R}$, which corresponds in this example to $\lambda \in S^1$, i.e., $|\lambda| = 1$. For more details, see [H], for example.

Following Gardner (see [G1] and [G2]), we define the periodic Evans function for our problem to be

$$D(\mu,k,\lambda) = \det\left(\mathbb{M}(\mu,k) - \lambda \mathbf{I}\right), \quad (\mu,k,\lambda) \in \mathbb{C} \times \mathbb{R} \times \mathbb{C}.$$

[5] Usually, one defines spectral stability as the absence of spectrum with positive real part. However, in our case the spectrum is symmetric about the imaginary axis, and hence spectral stability is equivalent with the spectrum being confined to the imaginary axis.

The complex constant λ is called the Floquet multiplier and is related to the class of admissible perturbations in (3.2). In particular, notice that $\lambda = 1$ corresponds to T-periodic perturbations of the background solution u. Clearly, $D(\mu, k, \lambda)$ is an entire function of μ and k for each fixed $\lambda \in \mathbb{C}$ since the coefficient matrix $\mathbf{H}(x, \mu, k)$ depends as such on μ and k. This allows an analytical characterization of the $L^2(\mathbb{R})$ spectrum of (3.2): the generalized spectral problem (3.2) has a nontrivial bounded solution for a given $k \in \mathbb{R}$ if and only if there exists a $\kappa \in \mathbb{R}$ such that

$$D(\mu, k, e^{i\kappa}) = 0.$$

In particular, $D(\mu, k, 1) = 0$ if and only if (3.2) has a nontrivial bounded solution in $L^2_{\text{per}}([0, T])$ for a given $k \in \mathbb{R}$. Moreover, the following property will be useful in the low frequency analysis conducted later in the paper.

LEMMA 1. *The function $D(\mu, k, \lambda)$ is an even function of both μ and k.*

Proof. Since the spectral problem $\left(\partial_x^2 \mathcal{L}[u] + \sigma k^2\right) v = \mu \partial_x v$ is invariant under the transformation $k \mapsto -k$, it follows that $D(\mu, k, \lambda)$ is an even function of k. To analyze the parity in μ, we write

$$\begin{aligned} D(\mu, k, \lambda) &= \det\left(\mathbb{M}(\mu, k) - \lambda \mathbf{I}\right) \\ &= \lambda^4 + a(\mu, k)\lambda^3 + b(\mu, k)\lambda^2 + c(\mu, k)\lambda + 1, \end{aligned}$$

where $a(\mu, k) = -\operatorname{tr}\left(\mathbb{M}(\mu, k)\right)$ and $b(\mu, k) = \frac{1}{2}\left(\operatorname{tr}(\mathbb{M}(\mu, k)^2) - \operatorname{tr}(\mathbb{M}(\mu, k))^2\right)$. Since the spectral problem is invariant under the transformation $x \to -x$ and $\mu \to -\mu$, it follows that the matricies $\mathbb{M}(\mu, k)$ and $\mathbb{M}(-\mu, k)^{-1}$ are similar, and hence a direct calculation yields

$$\begin{aligned} D(\mu, k, \lambda) &= \lambda^4 \det\left(\mathbb{M}(-\mu, k) - \frac{1}{\lambda}\mathbf{I}\right) \\ &= \lambda^4 + c(-\mu, k)\lambda^3 + b(-\mu, k)\lambda^2 + a(-\mu, k)\lambda + 1. \end{aligned}$$

It follows that $c(\mu, k) = a(-\mu, k)$ and $b(\mu, k) = b(-\mu, k)$, and hence

$$D(\mu, k, 1) = 2 + a(\mu, k) + a(-\mu, k) + \frac{b(\mu, k) + b(-\mu, k)}{2}.$$

Thus, $D(\mu, k, 1)$ is an even function of μ. $\square$

The goal of our analysis is to provide sufficient conditions to ensure that when $0 < |k| \ll 1$, the function $\mu \mapsto D(\mu, k, 1)$ has a nonzero real root, corresponding to an exponential instability of the underlying wave. To this end, we derive an orientation index by comparing the high frequency and low frequency (in μ) asymptotics of the function $D(\mu, k, 1)$. In particular, we will see that the sign of $D(\mu, k, 1)$ for large real μ equals the sign of σ. Thus, if the quantity $D(0, k, 1)$ has the opposite sign of σ, we can infer the existence of a $\mu^* \in \mathbb{R}^+$ such that $D(\mu^*, k, 1) = 0$, which implies exponential instability of the background solution. We begin by analyzing the high frequency behavior of the periodic Evans function.

3.1. High frequency limit. In this section, we study the large real μ behavior of the periodic Evans function $D(\mu, k, 1)$ when $k \neq 0$. To begin, rescale (3.2) with the change of variables $\tilde{x} = |\mu|^{1/3} x$ to obtain (after dropping the tildes) the spectral problem

$$\left(-\partial_x^4 - |\mu|^{-2/3}\partial_x^2\left(f'(u) + c\right) - \sigma k^2 |\mu|^{-4/3}\right) v = \partial_x v. \tag{3.3}$$

This can be rewritten as a first order system of the form

$$\mathbf{W}' = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{pmatrix}}_{\mathbf{H}_0(\mu)} \mathbf{W} + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \chi & A_1|\mu|^{-2/3} & A_2|\mu|^{-2/3} & 0 \end{pmatrix}}_{\mathbf{B}(\mu)} \mathbf{W}, \tag{3.4}$$

where $A_1 = -2f''(u)u_x$, $A_2 = -f'(u) + c$, and

$$\chi = \frac{1}{2}A_{1,x}|\mu|^{-2/3} - \sigma k^2|\mu|^{-4/3}.$$

On a heuristic level then, we expect that the monodromy operator for $\mu \gg 1$ will behave like

$$\mathbb{M}(\mu) \approx e^{\mathbf{H}_0|\mu|^{1/3}T},$$

and hence

$$D(\mu, k, 1) \approx \det\left(e^{\mathbf{H}_0|\mu|^{1/3}T} - \mathbf{I}\right).$$

However, the matrix $\mathbf{H}_0$ clearly has an eigenvalue of 0, and hence this heuristic argument leads us to expect that $D(\mu, k, 1) \to 0$ as $\mu \to +\infty$. From the point of view of an orientation index, this does not provide us with sufficient information: we must know what the limiting *sign* of the Evans function is. Thus, although we expect that $D(\mu, k, 1)$ vanishes in the limit as $\mu \to \infty$, we must analyze the situation more closely to determine if the *sign* of $D(\mu, k, 1)$ has a limiting value. This is the content of the following lemma.

LEMMA 2. *For $k \neq 0$, we have the high frequency limit*

$$\lim_{\mu\to\pm\infty} \operatorname{sgn}(D(\mu, k, 1)) = \operatorname{sgn}(\sigma).$$

Remark 1. Notice this result is somewhat unexpected due to the form of the rescaled equation (3.3) since the term σk^2 enters at only *second* order in $|\mu|^{-2/3}$. Thus, this is a *very* small term compared with the $\mathcal{O}(|\mu|^{-2/3})$ terms involved. However, the following proof will show that, upon averaging, the periodicity of the underlying solution implies cancelation of the lower order effects, and hence the asymptotic behavior for large μ must be determined by the higher order terms.

Proof. First, notice that by Lemma 1 it is enough to consider the limit as $\mu \to +\infty$ only. Consider the rescaled first order system (3.4) and define $\varepsilon = |\mu|^{-2/3} \ll 1$. We refer to the constant matrix $\mathbf{H}_0$ as the principle part and regard the matrix $\mathbf{B}$ as a matrix of error terms. We begin by diagonalizing the principle part. To this end, let $\lambda = \frac{1}{2}(1 + i\sqrt{3})$, and notice if we define the matrix

$$\mathbf{Q} := \begin{pmatrix} -1 & -1 & -1 & 1 \\ 1 & -\lambda & -\lambda^* & 0 \\ -1 & \lambda^* & \lambda & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

then a straightforward calculation yields

$$\mathbf{Q}^{-1}\mathbf{H_0}\mathbf{Q} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda^* & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $\Re\lambda = \Re\lambda^*$ is positive; hence the real parts of the diagonal entries of the stable, neutral, and unstable diagonal blocks of $\mathbf{Q}^{-1}\mathbf{H_0}\mathbf{Q}$ each have a spectral gap, one from the other. Using the change coordinates $\mathbf{W} = \mathbf{QY}$ in (3.4), we can consider the first order system

$$\mathbf{Y}' = \left[\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda^* & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \underbrace{\mathbf{Q}^{-1}\mathbf{B}\mathbf{Q}}_{\tilde{\mathbf{B}}}\right]\mathbf{Y}.$$

By a brief but tedious calculation, we see that the matrix $\tilde{\mathbf{B}}$ takes the block form

$$\tilde{\mathbf{B}} = \left(\begin{array}{ccc:c} & & & \frac{1}{3}\chi \\ & \mathcal{O}(\varepsilon) & & \frac{1}{3}\chi \\ & & & \frac{1}{3}\chi \\ \hdashline (A_1 - A_2)\varepsilon - \chi & (-\lambda A_1 + \lambda^* A_2)\varepsilon - \chi & (-\lambda^* A_1 + \lambda A_2)\varepsilon - \chi & \chi \end{array}\right)$$

where the upper left-hand block is 3×3 and the lower right-hand block is 1×1. Now define the T-periodic matrix valued function

$$\mathbf{S} = \mathbf{I}_4 + \varepsilon \left(\begin{array}{ccc:c} & 0 & & 0 \\ \hdashline -A_1 + A_2 + \frac{1}{2}A_{1,x} & -A_1 + \frac{\lambda^*}{\lambda}A_2 + \frac{1}{2\lambda}A_{1,x} & -A_1 + \frac{\lambda}{\lambda *}A_2 + \frac{1}{2\lambda *}A_{1,x} & 0 \end{array}\right),$$

where $\mathbf{I}_4$ is the standard 4×4 identity matrix and again the upper left-hand block is 3×3. Then another straightforward computation implies that

$$\mathbf{S}^{-1}\tilde{\mathbf{B}}\mathbf{S} = \left(\begin{array}{c:c} \mathcal{O}(\varepsilon) & \mathcal{O}(\varepsilon) \\ \hdashline \mathcal{O}(\varepsilon^3) & \frac{1}{2}A_{1,x}\varepsilon + \varepsilon^2\left(\frac{1}{2}A_1A_{1,x} - \sigma k^2\right) \end{array}\right)$$

where again the upper left-hand block is 3×3.

Now noticing that $\mathbf{S}' = \mathcal{O}(\varepsilon^{3/2})$ and that $\mathbf{S}$ is a $\mathcal{O}(\varepsilon)$ perturbation of the identity, it follows that making the variable coefficient change of variables $\mathbf{U} = \mathbf{SY}$ yields a first order system of the form

$$\mathbf{U}_x = \left[\mathbf{Q}^{-1}\mathbf{H}_0\mathbf{Q} + \left(\begin{array}{c:c} \mathcal{O}(\varepsilon) & \mathcal{O}(\varepsilon) \\ \hdashline \mathcal{O}(\varepsilon^{3/2}) & \frac{1}{2}A_{1,x}\varepsilon + \varepsilon^2\left(\frac{1}{2}A_1A_{1,x} - \sigma k^2\right) \end{array}\right)\right]\mathbf{U}.$$

In particular, we see that the resulting coefficient matrix is approximately block upper triangular with error of order $\mathcal{O}(\varepsilon^{3/2}) = \mathcal{O}(|\mu|^{-1})$. Applying Lemma A.1 from Appendix A (notice that here $\eta = 1$, $N = \mathcal{O}(\varepsilon)$, and $\delta = \mathcal{O}(\varepsilon^{3/2})$) and Remark 7 we find that there is a T-periodic change of coordinates $\mathbf{X} = \mathbf{ZU}$ of the form

$$\mathbf{Z} = \begin{pmatrix} I_3 & 0 \\ \Phi & 1 \end{pmatrix},$$

where $\Phi = O(\varepsilon^{3/2})$ is of dimension 1×3, taking the system to an exact upper block-triangular form with diagonal blocks

$$-1+\mathcal{O}(\varepsilon), \quad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^* \end{pmatrix} + O(\varepsilon), \quad \text{and } \frac{1}{2}A_{1,x}\varepsilon + \varepsilon^2\left(\frac{1}{2}A_1A_{1,x} - \sigma k^2\right) + \mathcal{O}(\varepsilon^{5/2}).$$

Finally, by the block-triangular form *plus periodicity of the coordinate changes* we may compute the periodic Evans function as the product of the periodic Evans function of the diagonal blocks of this transformed system, integrated over a period $\tilde{T} := T|\mu|^{1/3}$ going to infinity. The contribution from the stable block is[6] approximately $e^{-|\mu|^{1/3}T} - 1$ and so has sign -1. Similarly, the unstable block gives a positive sign. The third block, corresponding to the neutral block, gives approximately

$$\begin{aligned} \exp\left(\int_0^{|\mu|^{1/3}T}\left(\frac{1}{2}A_{1,x}\varepsilon + \varepsilon^2\left(\frac{1}{2}A_1A_{1,x} - \sigma k^2\right)\right)(s)ds\right) - 1 \\ = \exp\left(-\sigma k^2|\mu|^{-1}T\right) - 1 \\ \sim -\sigma k^2|\mu|^{-1} \end{aligned}$$

and hence has the opposite sign as the dispersion parameter σ. Notice, here we have used heavily the fact that the periodicity of the background solution implies

$$\int_0^{|\mu|^{1/3}T} A_{1,x}(x)dx = 0 \quad \text{and} \quad \int_0^{|\mu|^{1/3}T} A_1(x)A_{1,x}(x)dx = 0$$

and hence yields cancellation in averaging of the lower order terms. Combining these results, we find that the periodic Evans function has sign $\mathrm{sgn}(\sigma)$ for $k \neq 0$ and $\mu \gg 1$ as claimed. ∎

Notice that one could rework the above proof in the case where the background solution corresponds to a homoclinic orbit of the traveling wave ODE (2.1). As a result, Lemma 2 still holds in the solitary wave setting. This result seems to be new to the literature and may give valuable insight to the transverse instability analysis in the solitary wave case.

3.2. Low frequency analysis and instability. Now that we have a handle on the limiting sign of $D(\mu,k,1)$ as $\mu \to \pm\infty$, we turn our attention to determining the sign of the quantity $D(0,k,1)$ for $|k| \ll 1$. For, once we have this information, we see that the negativity of the orientation index

$$\sigma \cdot \mathrm{sgn}\left(D(0,k,1)\right)$$

provides a sufficient condition for transverse instability of the underlying periodic wave u. To this end, we utilize the methods of [J2], which build off of the methods of [BrJ], to derive an asymptotic expansion of the function $D(0,k,1)$ for $|k| \ll 1$. However, it should be pointed out that the analysis in this case is seemingly more delicate than that considered by [J2] due to the fact that the unperturbed spectral problem at $\mu = 0$, i.e.,

$$\partial_x^2 \mathcal{L}[u]v = 0, \tag{3.5}$$

[6]Notice each of the three subsystems gives real value, since they clearly have real coefficients.

which does not define a Hamiltonian equation. In particular, this equation is not reducible to quadrature, and hence one cannot use integrability of the corresponding traveling wave ODE

$$\left(-u_{xxx} - f(u)_x + cu_x\right)_x = 0$$

to construct a basis for the monodromy operator at $\mu = 0$, $k = 0$. However, one can use the integrability of the traveling wave ODE (2.1) for the gKdV equation to construct[7] three linearly independent solutions of (3.5), namely, the functions u_x, u_a, and u_E are easily seen to satisfy

$$\mathcal{L}[u]u_x = 0, \quad \mathcal{L}[u]u_E = 0, \quad \mathcal{L}[u]u_a = -1$$

(see [BrJ] for details) and hence provide a basis of solutions for the differential equation

$$\partial_x \mathcal{L}[u]v = 0.$$

Thus, we are missing one null direction of the (formal) operator $\partial_x^2 \mathcal{L}[u]$: we need a solution to the equation $\mathcal{L}[u]v = x$. However, notice that the functions u_x and u_E provide two linearly independent solutions of the differential equation[8] $\mathcal{L}[u]v = 0$, and hence one can use variation of parameters to solve the nonhomogeneous equation. After a straightforward calculation it is seen that

$$\phi(x) := \left(\int_0^x s u_E(s)ds\right) u_x(x) - \left(\int_0^x s u_s(s)ds\right) u_E(x)$$

is linearly independent from u_x, u_a, and u_E and satisfies $\mathcal{L}[u]\phi = x$, and hence we can use these four functions to construct the corresponding monodromy matrix at the origin. Using perturbation theory then, we should be able to determine how these functions bifurcate as k varies but remains very small and hence be able to determine a leading order expansion of the periodic Evans function near $k = 0$. This is the content of the following lemma.

LEMMA 3. *The following asymptotic relation holds in a neighborhood of $k = 0$:*

$$D(0,k,1) = -\left(PT - M^2\right)\{T, M\}_{a,E}\left(\sigma k^2\right)^2 + \mathcal{O}(k^6).$$

Proof. We begin by writing the linearized equation (3.2) as a first order system with coefficient matrix

$$\mathbf{H}(x,\mu,k) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\sigma k^2 - f'''(u)u_x^2 - f''(u)u_{xx} & -2f''(u)u_x - \mu & -f'(u) + c & 0 \end{pmatrix}.$$

We now define the matrix $\mathbf{W}(x,\mu,k)$ as the matrix solution of the first order system $Y' = \mathbf{H}(x,\mu,k)Y$ such that

$$\mathbf{W}(x,0,0) = \begin{pmatrix} u_x & u_a & u_E & \phi \\ u_{xx} & u_{ax} & u_{Ex} & \phi_x \\ u_{xxx} & u_{axx} & u_{Exx} & \phi_{xx} \\ u_{xxxx} & u_{axxx} & u_{Exxx} & \phi_{xxx} \end{pmatrix}, \tag{3.6}$$

[7]Notice that here we are considering the traveling wave ODE as a formal differential equation without any reference to boundary conditions.

[8]Here, again, we consider the formal operator $\mathcal{L}[u]$ without any reference to boundary conditions.

and we fix the initial condition $\mathbf{W}(0,\mu,k) = \mathbf{W}(0,0,0)$ for all $(\mu,k) \in \mathbb{C}\times\mathbb{R}$. Defining $\delta\,\mathbf{W}(\mu,k) = \mathbf{W}(x,\mu,k)\big|_{x=0}^{T}$, a straightforward calculation gives

$$\delta\,\mathbf{W}(0,0) = \begin{pmatrix} 0 & 0 & 0 & -\frac{\partial u_-}{\partial E}\int_0^T xu_x(x)dx \\ 0 & V'(u_-)T_a & V'(u_-)T_E & -V'(u_-)\int_0^T xu_E(x)dx \\ 0 & 0 & 0 & -T + V''(u_-)\frac{\partial u_-}{\partial E}\int_0^T xu_x(x)dx \\ 0 & -V'(u_-)V''(u_-)T_a & -V'(u_-)V''(u_-)T_E & V''(u_-)V'(u_-)\int_0^T xu_E(x)dx \end{pmatrix}.$$

Since $\delta\,\mathbf{W}(0,k)$ is analytic in k^2 by Lemma 1, it follows that

$$D(0,k,1) = \frac{\det(\delta\,\mathbf{W}(0,k))}{\det(\mathbf{W}(0,0,1))} = \mathcal{O}(k^4)$$

for $|k| \ll 1$. Moreover, since the first column of the matrix $\delta\,\mathbf{W}(0,k)$ is $\mathcal{O}(k^2)$, one can also easily see that the $\mathcal{O}(k^2)$ variation in the ϕ direction contributes to terms in $D(0,k,1)$ of order $\mathcal{O}(k^6)$ near $k=0$. Thus, in order to compute the $\mathcal{O}(k^4)$ variation in the Evans function, we need only compute the $\mathcal{O}(k^2)$ variation in the u_x, u_a, and u_E directions.[9]

Computing the necessary variations can be done using the variation of parameters formula. To this end, we define the vector solutions corresponding to u_x, u_a, u_E, and ϕ be given by Y_1, Y_2, Y_3, and Y_4, respectively, and define

$$\frac{\partial}{\partial k^2}Y_j(T,0,k)\big|_{k=0} = \mathbf{W}(T,0,0)\int_0^T \mathbf{W}(x,0,0)^{-1}\left(Y_j(x)\cdot e_j\right)e_4 dx,$$

where $\cdot$ represents the standard inner product on $\mathbb{R}^4$, e_j is the jth column of the identity matrix on $\mathbb{R}^4$, and $\mathbf{W}(T,0,0) = \mathbf{W}(0,0,0) + \delta\,\mathbf{W}(0,0)$, where

$$\mathbf{W}(0,0,0) = \begin{pmatrix} 0 & \frac{\partial u_-}{\partial a} & \frac{\partial u_-}{\partial E} & 0 \\ -V'(u_-) & 0 & 0 & 0 \\ 0 & 1 - V''(u_-)\frac{\partial u_-}{\partial a} & -V''(u_-)\frac{\partial u_-}{\partial E} & 0 \\ V''(u_-)V'(u_-) & 0 & 0 & -1 \end{pmatrix}.$$

Moreover, in Appendix B we will show that

$$\mathbf{W}(x,0,0)^{-1}e_4 = \begin{pmatrix} -\int_0^x\int_0^s u_E(z)dzds \\ -x \\ \int_0^x u(s)ds \\ -1 \end{pmatrix}, \tag{3.7}$$

which allows for a straightforward computation of the necessary variations. Thus, we have

$$\delta\,\mathbf{W}(0,k) = \delta\,\mathbf{W}(0,0) - \left(\sum_{j=1}^{4}\frac{\partial}{\partial k^2}Y_j(T,0,k)\big|_{k=0}\otimes e_j\right)\sigma k^2 + \mathcal{O}(k^4),$$

[9] Note that finding a useful expression for the $\mathcal{O}(k^2)$ variation in the ϕ direction would be a very daunting task, since one would have to apply a variation of parameters to solve a problem with nonhomogeneity which was constructed using a variation of parameters.

and since $\det(\mathbf{W}(0,0,0)) = 1$, it follows that

$$D(0,k,1) = \det\left(\delta\,\mathbf{W}(0,0) - \left(\sum_{j=1}^{3} \frac{\partial}{\partial k^2} Y_j(T,0,k)\big|_{k=0} \otimes e_j\right)\sigma k^2\right) + \mathcal{O}(k^6).$$

Elementary row operations allow one to simplify the matrix involved in the above determinant. Indeed, a large simplification is possible if one replaces the third row with the result from multiplying the first row by $V''(u_-)$ and adding it to the third row, and similarly replace the fourth row with $V''(u_-)$ times the second row plus the fourth row. In particular, these operations imply the fourth entry in the first column is of order $\mathcal{O}(k^4)$ and hence does not enter into the calculation. The resulting expressions are still too long to explicitly write out here, but can be easily handled by using a computer algebra system. Indeed, a direct calculation yields

$$D(0,k,1) = \left(Q_1 \frac{\partial u_-}{\partial E} + Q_2 \frac{\partial u_-}{\partial a}\right) V'(u_-)\{T,M\}_{a,E}(\sigma k^2)^2 + \mathcal{O}(k^6),$$

where $Q_1 = (M^2 - PT - MTu_- + T^2u_-^2)$ and $Q_2 = (MT - T^2u_-)$. Finally, by noticing from (2.2) that

$$V'(u_-)\frac{\partial u_-}{\partial E} = 1 \quad \text{and} \quad V'(u_-)\frac{\partial u_-}{\partial a} = u_-,$$

we have

$$\left(Q_1 \frac{\partial u_-}{\partial E} + Q_2 \frac{\partial u_-}{\partial a}\right) V'(u_-) = M^2 - PT,$$

which completes the proof. $\square$

Remark 2. In [BrJ], Bronski and Johnson derive the low frequency expansion for the Evans function $D_{gKdV}(\mu,\kappa)$ for the gKdV equation (1.3) in the spectral parameter μ. In particular, it was shown that

$$D_{gKdV}(\mu,1) = -\frac{1}{2}\det\left(\frac{\partial(T,M,P)}{\partial(a,E,c)}\right)\mu^3 + \mathcal{O}(|\mu|^4),$$

and hence an index which detects exponential instabilities to coperiodic perturbations was derived by comparing the sign of the above Jacobian determinant to the limiting behavior for $\mu \to +\infty$. We believe that it would be interesting and beneficial to derive the corresponding low frequency expansion for the Evans function $D(\mu,k,\kappa)$ considered in this paper, as it may give a better understanding of the connection (if there is any) between the generation of transverse instabilities and uni-directional instabilities. Indeed, such analysis could serve as a starting point for a corresponding transverse *stability* analysis of periodic waves. While it seems natural that the above three-by-three Jacobian determinant should control the (spectral) low frequency limit, it is not clear how the degeneracy of the gKP equation (corresponding to the extra x-spatial derivative) affects the situation. In particular, the required calculations seem to be considerably more complicated than those considered in Lemma 3, and a useful identification of the leading order behavior for $|\mu| \ll 1$ is yet to be obtained.

Combining Lemmas 2 and 3, we have a sufficient condition for exponential instability of a periodic traveling wave of the gKdV to long wavelength transverse perturbations in the gKP equation. This is the content of our main theorem, which we now state.

THEOREM 1. *Let $u = u(\cdot; a, E, c)$ be a periodic traveling wave of the gKdV equation* (1.3)*. Then u is spectrally unstable to transverse perturbations in the gKP equation* (1.4) *if the product $\sigma \cdot \{T, M\}_{a,E}$ is positive.*

Proof. Clearly the negativity of the orientation index

$$\operatorname{sgn}(D(0,k,1)) \cdot \lim_{\mu \to +\infty} \operatorname{sgn}(D(\mu,k,1))$$

for $0 < |k| \ll 1$ implies the desired instability. Since $PT - M^2 > 0$ by Jensen's inequality, the result follows by Lemmas 2 and 3. $\square$

Remark 3. Notice the instability detected in Theorem 1 is that of spectral transverse instabilities to perturbations which are coperiodic in the x-direction with low frequency oscillations in the transverse (y) direction. In the solitary wave case, a general criterion for spectral transverse instability was recently provided by Rousset ant Tzvetkov [RT1]. While it seems plausible that such a criterion may exist when the underlying wave is spatially periodic, we must note that the detected instability would not be to *low* frequency oscillations in the transverse direction. Indeed, in [RT1] the transverse frequency must be large enough that the kernel of a particular linear operator be simple. Once such a frequency $k_0 \neq 0$ is found, an implicit function-type argument is used to prove the existence of a spectral curve $\mu \to (k(\mu), \mu)$ in a neighborhood of $\mu = 0$ with $(k(0), 0) = (k_0, 0)$, thus proving spectral instability. In the present work, however, we are proving the existence of a map $k \to \mu(k)$ defined for $|k| \ll 1$ such that $\mu(k)$ bounded away from zero for all $k \neq 0$ and such that $\mu(k)$ is an eigenvalue of (3.2) for the associated small transverse frequency k.

As previously noted, Theorem 1 is only of interest when the underlying periodic wave is a stable solution of the corresponding one-dimensional problem. In [BrJK], the nonlinear (orbital) stability of such solutions to periodic perturbations in the gKdV equation was studied, and the Jacobian $\{T, M\}_{a,E}$ was seen to play a significant role. Therein, it was shown that such waves can be nonlinearly stable to such perturbations regardless of the sign of $\{T, M\}_{a,E}$, assuming certain conditions on the perturbation and other geometric quantities related to the underlying wave and the conserved quantities of the PDE flow. Theorem 1, however, demonstrates the direct influence the sign of this Jacobian determinant has on the stability to perturbations in higher-dimensional models (see also the recent work of Johnson [J2] concerning the transverse instability of periodic gKdV waves in the generalized Zakharov–Kuznetsov equations where the same Jacobian was seen to control the low frequency behavior of the corresponding Evans function). In particular, we immediately have the following interesting (and seemingly unexpected) corollary.

COROLLARY 1. *A periodic traveling wave of the gKdV for which $\{T, M\}_{a,E} \neq 0$ can never be spectrally stable to transverse perturbations in the gKP equation for both signs of dispersion.*

Remark 4. It should be noted that, unlike the ODE case, there are no general theorems ensuring that spectral instability implies nonlinear instability. However, in the recent work [RT3] it was shown in the solitary wave context that, indeed, spectral transverse instability of KdV waves in the KP-I equation (as described in the introduction) can be converted to a nonlinear instability result. It seems plausible that the methods utilized could apply in our case in order to convert Theorem 1 into a nonlinear instability result (at least in the case of periodic transverse perturbations). This would be an interesting direction for future investigation.

We now point out several corollaries of Theorem 1. In the case of the KdV equation (1.1), it is known that all periodic traveling wave solutions are both spectrally stable to localized perturbations (see [BD]) and nonlinearly (orbitally) stable to coperiodic perturbations (see [BrJK] or [J1]). In particular, the transverse stability of such solutions in the KP equation (1.2) is of interest in this case. Moreover, it was shown in [BrJK] that the Jacobian $\{T, M\}_{a,E}$ can be expressed as

$$\{T, M\}_{a,E} = \frac{-T^2\, V'\left(\frac{M}{T}\right)}{12\ \mathrm{disc}\,(E - V(\cdot; a, c))},$$

where $\mathrm{disc}(R(\cdot))$ represents the discriminant of the polynomial R. Since V' is clearly strictly convex in this case, it follows by Jensen's inequality that

$$V'\left(\frac{M}{T}\right) < \frac{1}{T}\int_0^T V'(u(x))dx = 0.$$

For an alternate proof of this fact, see [J2]. Moreover, notice that for any $(a, E, c) \in \mathbb{R}^3$ for which (2.1) admits a periodic solution of the KdV the equation $E = V(u; a, c)$ has three solutions in u, and hence the discriminant must be positive. Therefore, we have that $\{T, M\}_{a,E} > 0$ for all periodic traveling wave solutions of the KdV equation. This proves the following corollary of Theorem 1.

Corollary 2. *All periodic traveling wave solutions of the KdV are unstable to long wavelength transverse perturbations in the KP equation when $\sigma > 0$.*

It is interesting to note that it is known that all solitary wave solutions of the KdV are transversely unstable in the KP equation when $\sigma > 0$. Thus, Corollary 2 seems to be somewhat expected. Moreover, it turns out that the Galilean invariance of the KdV implies we can always choose $a = 0$, and hence (up to translation) the periodic traveling waves of the KdV form only a *two* parameter family of solutions. This family can be expressed explicitly in terms of the Jacobi elliptic function as

$$u(x,t) = u_0 + 12k^2\kappa^2\,\mathrm{cn}^2\left(\kappa\left(x + \left(8k^2\kappa^2 - 4\kappa^2 + u_0\right)t\right), k\right),$$

where u_0 is an arbitrary parameter (taking the role of E) and k is the elliptic modulus. We refer to such a solution as a cnoidal wave solution of the KdV. As a result of Corollary 2, it follows that all cnoidal wave solutions of the KdV are unstable to long wavelength transverse perturbations in the KP-I equation.

We now move on to consider periodic traveling wave solutions of the focusing mKdV equation

$$u_t = u_{xxx} + u^2 u_x$$

with positive wave speed $c > 0$. When $a = 0$, the corresponding traveling wave ODE admits two distinct classes of periodic solutions: when $E > 0$ the wave can again be expressed in terms of the Jacobi elliptic function cn, while when $E < 0$ there either exists no solution (if $|E|$ is sufficiently large) or the solution can be expressed in terms of the Jacobi elliptic function dn and hence represents a dnoidal wave. In the recent work of [BrJK], it was shown that both sets of solutions are nonlinearly (orbitally) stable to coperiodic perturbations, although the cnoidal solutions of sufficiently long wavelength were shown to be (spectrally) unstable to periodic perturbations of large period[10] (see also [DK]). Moreover, the sign of the Jacobian $\{T, M\}_{a,E}$ was analyzed

[10] However, cnoidal waves of smaller period seem spectrally stable to perturbations of sufficiently large period.

for both the cnoidal and dnoidal wave solutions of the focusing mKdV[11] and was seen to be positive for all dnoidal solutions and negative for all cnoidal solutions. As a result, we have the following corollary of Theorem 1.

COROLLARY 3. *All cnoidal wave solutions of the focusing mKdV equation are unstable to long wavelength transverse perturbations in the focusing mKP equation with* $\sigma < 0$*, while the dnoidal wave solutions are unstable to such perturbations when* $\sigma > 0$.

Continuing, one can use the general elliptic function calculations of [BrJK] in the case of a power-law nonlinearity $f(u) = u^{p+1}$, $p \in \mathbb{N}$, to determine the sign of the Jacobian $\{T, M\}_{a,E}$ for any periodic traveling wave solution of (1.3) in terms of moments of the background solution u with respect to the density

$$\frac{1}{\sqrt{E - V(\cdot; a, c)}}.$$

As such, Theorem 1 can be utilized to provide a transverse instability result for such power-law nonlinearities. In other cases, it seems that one must (in general) resort to numerical methods to approximate $\{T, M\}_{a,E}$.

4. Conclusions and discussion. In this paper, we analyzed the spectral instability of a periodic traveling wave solutions of the generalized KdV equation to long wavelength transverse perturbations in the generalized KP equation. In particular, we constructed a seemingly nonstandard orientation index by comparing the low and high frequency behavior of the periodic Evans function when the transverse wave number k is nonzero. We found that in the high frequency limit, the Evans function $D(\mu, k, 1)$ converged to zero as $\mu \to \pm\infty$, which is insufficient to conclude an instability theory. However, after taking into account higher order effects, it was found that by the periodicity of the underlying wave and the resulting cancelation in averaging procedures that the Evans function for nonzero transverse wave numbers favors a particular *sign* as $\mu \to \pm\infty$, which is determined precisely by the dispersion parameter σ; such a phenomenon seems to be new in the literature. Thus, an instability index follows by comparing the sign of σ with the value $D(0, k, 1)$ for $k \neq 0$. Utilizing the methods of [BrJ] and [J2] then, we were able to explicitly compute the leading order variation of the function $D(0, k, 1)$ in k in terms of a Jacobian from the traveling wave parameters to the period and mass of the background solution. This Jacobian was shown to be (generically) nonzero in the physically important cases of the KdV and mKdV equations, and their resulting signs were inferred from the recent work of [BrJK], immediately yielding instability results in these cases.

It is interesting to note that the Jacobian arising in the low frequency expansion of the periodic Evans function has already been seen to hold vital information concerning the stability of the periodic traveling wave solutions of the gKdV. Indeed, in [BrJK] and [J1] this Jacobian arose naturally in the nonlinear stability analysis of such solutions to periodic perturbations in the gKdV equation, while in [J2] it was seen again to control the low frequency behavior of the periodic Evans function when considering the spectral instability of a periodic gKdV wave to long wavelength transverse perturbations in the generalized Zakharov–Kuznetsov equation (which also arises in plasma physics). Thus, it would be very interesting to better understand the phys-

[11] In fact, the situation was analyzed without the restriction of $a = 0$, in which case all periodic traveling wave solutions of the focusing mKdV cannot be expressed simply in terms of a Jacobi elliptic function. However, we consider only the case $a = 0$ here for simplicity.

ical meaning of the Jacobian $\{T, M\}_{a,E}$, as this may better illuminate the stability theories described above.

Another interesting direction would be to complement the transverse *instability* analysis in this paper with a corresponding *stability* theory. That is, to derive sufficient conditions to guarantee that a periodic traveling wave solution of the gKdV is transversely stable to perturbations in the gKP equation. While this certainly may be possible using the Evans function techniques of this paper, much more delicate analysis is needed. In the context of the KdV equation, one may be able to use the integrable structure of the KP-II equation to prove transverse spectral stability. Such techniques are prevalent in solitary wave theory and have recently been employed in the periodic wave setting to prove spectral stability to localized perturbations for several model equations (see [BD], [BDN], and [NB]). When this integrable structure does not exist, however, it may be more natural to consider a variational characterization of the stability problem and extend the methods of [GSS1], [GSS2], [J1], and [BrJK].

We also note that, after the completion of this work, our attention was brought to the recent work of Rousset and Tzvetkov [RT1] in which the authors considered the transverse spectral instability of KdV solitary waves to perturbations in the KP-I equation. In contrast to the ODE techniques utilized in this paper, the authors use variational techniques to present a rather elegant and simple approach relying only on properties of the differential operators involved which are rather easy to check (due to the self-adjointness of the operators involved). We believe it would be interesting as a future direction of study to see if these techniques could apply in the case where the underlying wave is spatially periodic and to compare the results to those derived in this paper. Such a comparison would hopefully help illuminate the mechanism behind the instability. On the other hand, the ODE techniques used in this paper are quite robust, allowing for straightforward numerical implementation (plotting Evans curves and computing winding numbers) and applying not only to equations with a Hamiltonian-like structure but also to more complicated situations arising in the context of systems of nonlinear conservation laws. Moreover, the degeneracy in the high frequency analysis is of independent interest, adding new techniques to the tool box in the study of nonlinear dispersive waves utilizing asymptotic tracking/reduction results familiar from shock wave analysis in the conservation law setting.

Finally, as pointed out in the text, the high frequency analysis conducted in this paper translates directly to the solitary wave setting and hence opens the door to an analogous instability theory using Evans function techniques. In fact, the high frequency limit seems easier to discern in the solitary wave case as one does not have to deal with averaging effects of the coefficient functions. While we have not yet conducted the relevant low frequency analysis, this could provide valuable insights concerning the transverse instability of a solitary traveling wave of the gKdV in the gKP equation.

Appendix A. A block-triangular tracking lemma. Consider an approximately block-triangular system

$$W' = A^p(x)W := \begin{pmatrix} M_1 & N \\ \delta\Theta & M_2 \end{pmatrix}(x,p)W, \tag{A.1}$$

where Θ is a uniformly bounded matrix, $\delta(x)$ scalar, and p a vector of parameters satisfying a pointwise spectral gap condition

$$\min \sigma(\Re M_1) - \max \sigma(\Re M_2) \geq \eta(x) > 0 \text{ for all } x. \tag{A.2}$$

(Here as usual $\Re N := (1/2)(N + N^*)$ denotes the "real" or symmetric part of N.) Then we have the following block-triangular version of the *tracking/reduction lemma* of [MaZ3, PZ]. For related results, see [HLZ].

LEMMA A.1. *Consider a system* (A.1) *under the gap assumption* (A.2), *with* Θ *uniformly bounded and* $\eta \in L^1_{\text{loc}}$. *If* $\sup(\delta/\eta)(x)$ *is sufficiently small, then there exists a unique bounded linear transformation*

$$S = \begin{pmatrix} I & 0 \\ \Phi & I \end{pmatrix}, \tag{A.3}$$

possessing the same regularity with respect to p *as do coefficients* M_j *and* N, *such that the change of coordinates* $W = SZ$ *converts the approximately triangular system* (A.1) *to an exactly block-triangular system*

$$Z' = \tilde{A}^p(x)Z := \begin{pmatrix} \tilde{M}_1 & \tilde{N} \\ 0 & \tilde{M}_2 \end{pmatrix}(x,p)Z, \tag{A.4}$$

where

$$\tilde{M}_1 := M_1 + \Phi N, \qquad \tilde{M}_2 := M_2 - \Phi N, \qquad \tilde{N} := N, \tag{A.5}$$

with

$$\sup |\Phi| \le C \sup(\delta/\eta)$$

and

$$|\Phi(x)| \le C \int_x^{+\infty} e^{\int_y^x \eta(z)dz} \delta(y)dy, \qquad |\Phi(x)| \le C \int_{-\infty}^x e^{\int_y^x -\eta(z)dz} \delta(y)dy, \tag{A.6}$$

where the constant C *depends only on the size of* Θ.

Proof. By the change of coordinates $x \to \tilde{x}$, $\delta \to \tilde{\delta} := \delta/\eta$ with $d\tilde{x}/dx = \eta(x)$, we may reduce to the case $\eta \equiv \text{constant} = 1$ treated in [MaZ3]. Dropping tildes, we find by direct computation that $Z = S^{-1}W$ satisfies (A.4) for S of form (A.3) if and only if (A.5) and

$$\Phi' = (M_2\Phi - \Phi M_1) + Q(\Phi),$$

where Q is the quadratic matrix polynomial $Q(\Phi) := \delta\Theta - \Phi N \Phi$. Viewed as a vector equation, this has the form

$$\Phi' = \mathcal{M}\Phi + Q(\Phi),$$

with linear operator $\mathcal{M}\Phi := M_2\Phi - \Phi M_1$. Note that a basis of solutions of the decoupled equation $\Phi' = \mathcal{M}\Phi$ may be obtained as the tensor product $\Phi = \phi\tilde{\phi}^*$ of bases of solutions of $\phi' = M_2\phi$ and $\tilde{\phi}' = -M_1^*\tilde{\phi}$, whence we obtain from (A.2) that

$$e^{\mathcal{M}z} \le Ce^{-\eta z}, \quad \text{for } \; z > 0, \tag{A.7}$$

or uniform exponentially decay in the forward direction.

Thus, assuming only that Φ is bounded at $-\infty$, we obtain by Duhamel's principle the integral fixed-point equation

$$\Phi(x) = \mathcal{T}\Phi(x) := \int_{-\infty}^x e^{\mathcal{M}(x-y)} Q(\Phi)(y)\, dy. \tag{A.8}$$

Using (A.7), we find that $\mathcal{T}$ is a contraction of order $O(\delta/\eta)$ for Φ on a ball of radius of the same order; hence (A.8) determines a unique solution for δ/η sufficiently small, which, moreover, is order δ/η as claimed. (Here, we are using the normalization $\eta = 1$.) Finally, substituting $Q(\Phi) = O(\delta + |\Phi|^2) = O(\delta)$ in (A.8), we obtain

$$|\Phi(x)| \leq C \int_{-\infty}^{x} e^{\eta(x-y)} \delta(y)\, dy$$

in $\tilde{x}$ coordinates or, in the original x-coordinates, (A.6). Regularity with respect to parameters is inherited as usual through the fixed-point construction via the implicit function theorem. $\square$

Remark 5. Although we do not use it here, an important observation of the above proof is that for η constant and δ decaying at an exponential rate strictly slower than $e^{-\eta x}$ as $x \to +\infty$, we find from (A.6) that $\Phi(x)$ decays like δ/η as $x \to +\infty$, while if $\delta(x)$ merely decays monotonically as $x \to -\infty$, we find that $\Phi(x)$ decays like (δ/η) as $x \to -\infty$.

Remark 6. Though we do not use it here, an important observation of [MaZ3, PZ] is that hypothesis (A.2) of Lemma A.1 may be weakened to

$$\min \sigma(\Re M_1^\varepsilon) - \max \sigma(\Re M_2^\varepsilon) \geq \eta(x) + \alpha(x,p) > 0$$

with no change in the conclusions for any α satisfying a uniform L^1 bound $|\alpha(\cdot,p)|_{L^1} \leq C_1$. (Substitute $e^{\mathcal{M}x} \leq Ce^{C_1} e^{-\eta z}$ for (A.7), with no other change in the proof.) This allows us to neglect commutator terms in some of the more delicate applications of tracking: for example, the high frequency analysis of [MaZ3].

Remark 7. In the special case where A^p is T-periodic in x, we obtain by uniqueness that Φ is T-periodic in x as well.

Appendix B. Variation of parameters calculation. In this appendix our goal is to justify (3.7), which was seen to be a crucial step in the low frequency analysis of section 3. For brevity, however, we will consider only the most difficult case. To begin, define the scalar valued function $A_0(x) := e_1^\dagger\, \mathbf{W}(x,0,0)^{-1} e_4$, where $\mathbf{W}(x,0,0)$ is defined in (3.6) and $\dagger$ represents the vector adjoint, and note from (3.7) that we wish to prove $A_0 = -\int_0^x \int_0^s u_E(z)dzds$. By definition, we see that

$$A_0 = (u_{axx}u_{Ex} - u_{ax}u_{Exx})\,\phi + (u_a u_{Exx} - u_{axx}u_E)\,\phi_x + (u_{ax}u_E - u_a u_{Ex})\,\phi_{xx}.$$

Using (2.2), we have that

$$\begin{aligned} u_a u_{Exx} - u_{axx}u - E &= -u_a V''(u) u_E + u_E\,(V''(u)u_a - 1) \\ &= -u_E. \end{aligned}$$

Noting that $\partial_x\,(u_a u_{Ex} - u_{ax}u_E) = u_a u_{Exx} - u_{axx}u_E$, it follows that

$$u_{ax}u_E - u_a u_{Ex} = \int_0^x u_E(s)ds.$$

Moreover, using (2.2) along with the fact that

$$u_x\,(u_{ax}u_E - u_a u_{Ex}) = uu_E - u_a,$$

it follows that

$$\begin{aligned} u_x\,(u_{ax}u_{Exx} - u_{axx}u_{Ex}) &= -u_{Ex}u_x - V''(u)\,(uu_E - u_a) \\ &= -u_{Ex}u_x - V''(u)u_x\,(u_{ax}u_E - u_a u_{Ex}) \\ &= -u_{Ex}u_x - V''(u)u_x \int_0^x u_E(s)ds. \end{aligned}$$

Therefore, using the above equalities along with the definition of the function ϕ, we have

$$\begin{aligned}A_0 =& - x\int_0^x u_E(s)ds\\&+\left(\left(u_{Ex}+V''(u)\int_0^x u_E(s)ds\right)u_x - u_Eu_{xx}+u_{xxx}\int_0^x u_E(s)ds\right)\int_0^x su_E(s)ds\\&+\left(-\left(u_{Ex}+V''(u)\int_0^x u_E(s)ds\right)u_E - u_Eu_{Ex}\right.\\&\qquad\left.+u_{Exx}\int_0^x u_E(s)ds\right)\int_0^x su_s(s)ds.\end{aligned}$$

Now as above one can show that

$$\begin{aligned}&\left(u_{Ex}+V''(u)\int_0^x u_E(s)ds\right)u_x - u_Eu_{xx}+u_{xxx}\int_0^x u_E(s)ds\\&\qquad\qquad= u_{Ex}u_x - u_Eu_{xx} + (V''(u)u_x+u_{xxx})\int_0^x u_E(s)ds\\&\qquad\qquad= 1\end{aligned}$$

since $u_{xxx} = -V''(u)u_x$ by (1.3). Similarly, it follows that

$$-\left(u_{Ex}+V''(u)\int_0^x u_E(s)ds\right)u_E - u_Eu_{Ex}+u_{Exx}\int_0^x u_E(s)ds = 0,$$

and hence

$$\begin{aligned}A_0(x) &= -x\int_0^x u_E(s)ds + \int_0^x su_E(s)ds\\&= \int_0^x (s-x)u_E(s)ds\\&= -\int_0^x\int_0^s u_E(z)dzds\end{aligned}$$

as claimed. The rest of the derivation of (3.7) is handled similarly, although the necessary calculations are considerably simpler.

Acknowledgment. Thanks to Jared Bronski, Björn Sandstede, Todd Kapitula, and Bernard Deconinck for many useful conversations regarding the early stages of this work. Their comments and suggestions were invaluable to this project. We would also like to thank a referee for bringing our attention to the work of Rousset and Tzvetkov and for several helpful comments regarding the structure of the KP equations.

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