

## NONLINEAR STABILITY OF VISCOUS ROLL WAVES\*

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**Abstract.** Extending results of Oh and Zumbrun and of Johnson and Zumbrun for parabolic conservation laws, we show that spectral stability implies nonlinear stability for spatially periodic viscous roll wave solutions of the one-dimensional St. Venant equations for shallow water flow down an inclined ramp. The main new issues to be overcome are incomplete parabolicity and the nonconservative form of the equations, which lead to undifferentiated quadratic source terms that cannot be handled using the estimates of the conservative case. The first is resolved by treating the equations in the more favorable Lagrangian coordinates, for which one can obtain large-amplitude nonlinear damping estimates similar to those carried out by Mascia and Zumbrun in the related shock wave case, assuming only symmetrizability of the hyperbolic part. The second is resolved by the observation that, similarly as in the relaxation and detonation cases, sources occurring in nonconservative components experience decay that is greater than expected, comparable to that experienced by a differentiated source.

**Key words.** roll waves, St. Venant equations, modulational stability

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**1. Introduction.** Roll waves are a well-known hydrodynamic instability occurring in shallow water flow down an inclined ramp, generated by competition between gravitational force and friction along the bottom. These can be modeled as periodic traveling-wave solutions of the St. Venant equations for shallow water flow, which take the form of hyperbolic or parabolic balance laws; see [4, 17, 18] for detailed discussions of existence in the inviscid and viscous cases.

The spectral and linear stability of roll waves has been studied for the inviscid St. Venant equations in [17] and the viscous St. Venant equations in [18]. However, up to now, the relation between spectral, linearized, and nonlinear stability has remained an outstanding open question. In this paper, extending recent results of [23, 8, 9] in the related conservation law case, we settle this question by showing that *spectral implies linearized and nonlinear stability*.

This opens the way to rigorous numerical and analytical exploration of stability of roll waves and related phenomena via the associated eigenvalue ODE, a standard and numerically and analytically well-conditioned problem. At the same time, it gives a particularly interesting application of the techniques of [23, 8, 9]. In particular, roll waves, by numerical and experimental observation, appear likely to be stable, at least in some regimes. In the parabolic conservation law case, by contrast, periodic waves so far appear typically to be unstable [20].

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**1.1. Equations and assumptions.** Consider the one-dimensional St. Venant equations approximating shallow water flow on an inclined ramp:

$$(1.1) \quad \begin{aligned} h_t + (hu)_x &= 0, \\ (hu)_t + (h^2/2F + hu^2)_x &= h - u^2 + \nu(hu_x)_x, \end{aligned}$$

where  $h$  represents height of the fluid;  $u$  is the velocity average with respect to height;  $F$  is the Froude number, which here is the square of the ratio between speed of the fluid and speed of gravity waves;  $\nu = \text{Re}^{-1}$  is a nondimensional viscosity equal to the inverse of the Reynolds number; the term  $u^2$  models turbulent friction along the bottom; and the coordinate  $x$  measures longitudinal distance along the ramp.

In Lagrangian coordinates, these appear as

$$(1.2) \quad \begin{aligned} \tau_t - u_x &= 0, \\ u_t + ((2F)^{-1}\tau^{-2})_x &= 1 - \tau u^2 + \nu(\tau^{-2}u_x)_x, \end{aligned}$$

where  $\tau := h^{-1}$  and  $x$  now denotes a Lagrangian marker rather than a physical location. We will work with this form of the equations, as it is more convenient for our analysis in several ways. (Indeed, for the large-amplitude damping estimates of section 4.4, it appears to be essential; see Remark 10.)

Denoting  $U := (\tau, u)$ , consider a spatially periodic traveling-wave solution

$$(1.3) \quad U = \bar{U}(x - ct)$$

of (1.2) of period  $X$  and wavespeed  $c$  satisfying the traveling-wave ODE

$$(1.4) \quad \begin{aligned} -c\tau' - u' &= 0, \\ -cu' + ((2F)^{-1}\tau^{-2})' &= 1 - \tau u^2 + \nu(\tau^{-2}u')'. \end{aligned}$$

Integrating the first equation of (1.4) and solving for  $u = u(\tau) := q - c\tau$ , where  $q$  is the resulting constant of integration, we obtain a second-order scalar profile equation in  $\tau$  alone:

$$(1.5) \quad c^2\tau' + ((2F)^{-1}\tau^{-2})' = 1 - \tau(q - c\tau)^2 - c\nu(\tau^{-2}\tau')'.$$

Note that nontrivial periodic solutions of speed  $c = 0$  do not exist in Lagrangian coordinates, as this would imply  $u \equiv q$ , and (1.5) would reduce to a scalar first-order equation

$$(1.6) \quad \tau' = F\tau^3(\tau q^2 - 1),$$

which since it is scalar first-order has no nontrivial periodic solutions, even degenerate ones (e.g., homoclinic or heteroclinic cycles) that might arise in the singular  $c \rightarrow 0$  limit. Rather, there appears to be a Hopf bifurcation as  $c$  approaches some minimum speed for which periodics exist; see [18, section 4.1 and Figure 1, section 4.2.3].

It follows then that periodic solutions of (1.5) correspond to values  $(X, c, q, b) \in \mathbb{R}^5$ , where  $X$ ,  $c$ , and  $q$  denote period, speed, and constant of integration, and  $b = (b_1, b_2)$  denotes the values of  $(\tau, \tau')$  at  $x = 0$ , such that the values of  $(\tau, \tau')$  at  $x = X$  of the solution of (1.5) are equal to the initial values  $(b_1, b_2)$ .

Following [29, 22, 23, 8, 9], we assume the following:

(H1)  $\bar{\tau} > 0$ , so that all terms in (1.2) are  $C^{K+1}$ ,  $K \geq 3$ .

(H2) The map  $H : \mathbb{R}^5 \rightarrow \mathbb{R}^2$  taking

$$(X, c, q, b) \mapsto (\tau, \tau')(X, c, b; X) - b$$

is full rank at  $(\bar{X}, \bar{c}, \bar{b})$ , where  $(\tau, \tau')(\cdot; \cdot)$  is the solution operator of (1.5).

By the implicit function theorem, conditions (H1)–(H2) imply that the set of periodic solutions in the vicinity of  $\bar{U}$  form a smooth three-dimensional manifold

$$(1.7) \quad \{\bar{U}^\beta(x - \alpha - c(\beta)t)\}, \text{ with } \alpha \in \mathbb{R}, \beta \in \mathbb{R}^2.$$

*Remark 1.* The transversality condition (H2) could be replaced by the more general assumption that the set of periodic solutions in the vicinity of  $\bar{U}$  form a smooth three-dimensional manifold (1.7). However, it is readily seen in this context that (H2) is then implied by the spectral stability condition (D3) of section 1.1.2; that is, transversality is necessary for our notion of spectral, or Evans, stability. This situation is reminiscent of that of the viscous shock case; see, for example, [36, section 1.2.3] or [15, 32].

*Remark 2.* Note that (1.2) is of  $2 \times 2$  viscous relaxation type,

$$(1.8) \quad U_t + f(U)_x - \nu(B(U)U_x)_x = \begin{pmatrix} 0 \\ q(U) \end{pmatrix}, \quad q(U) = 1 - \tau u^2,$$

where  $q_u = -2u\tau < 0$  for solutions  $u > 0$  progressing down the ramp. Thus, constant solutions are stable as long as the subcharacteristic condition  $|\frac{u^3}{2}| < |\frac{u^3}{\sqrt{F}}|$  is satisfied, or  $F < 4$ . When the subcharacteristic condition is violated, roll waves appear through Hopf bifurcation as parameters are varied through the minimum speed  $c_{\min} = \frac{1}{\sqrt{F}\tau_0^3}$ ; see Appendix C. For  $\nu = 0$ , violation of the subcharacteristic condition is associated with subshocks and the appearance of discontinuous roll waves observed by Dressler [4]; see [7] for a related, more general discussion.

*Remark 3.* The limit  $\nu \rightarrow 0$  represents an interesting singular perturbation problem in which the structure of the profile equations simplifies, decoupling into fast and slow scalar components, and converging to inviscid Dressler waves [4, 17] in an appropriate regime [18]. This would be an interesting setting in which to investigate the associated spectral stability problem. Another interesting limit is Hopf bifurcation from the constant solution occurring at minimum speed of existence [18], treated here in Appendix C; see Remark 12.

**1.1.1. Linearized equations.** Making the change of variables  $x \rightarrow x - ct$  to co-moving coordinates, we convert (1.2) to

$$(1.9) \quad \begin{aligned} \tau_t - c\tau_x - u_x &= 0, \\ u_t - cu_x + ((2F)^{-1}\tau^{-2})_x &= 1 - \tau u^2 + \nu(\tau^{-2}u_x)_x, \end{aligned}$$

and we convert the traveling-wave solution to a stationary solution  $U = \bar{U}(x)$  convenient for stability analysis.

Writing (1.9) in abstract form,

$$(1.10) \quad U_t + f(U)_x = (B(U)U_x)_x + g(U),$$

and linearizing (1.9) about  $\bar{U}(\cdot)$ , we obtain

$$(1.11) \quad v_t = Lv := (\partial_x B \partial_x - \partial_x A + C)v,$$

where the coefficients

$$(1.12) \quad \begin{aligned} A &:= df(\bar{u}) - (dB(\bar{u})(\cdot))\bar{u}_x = \begin{pmatrix} -c & -1 \\ -\bar{\tau}^{-3}(F^{-1} - 2\nu\bar{u}_x) & -c \end{pmatrix}, \\ B &:= B(\bar{u}) = \begin{pmatrix} 0 & 0 \\ 0 & \nu\bar{\tau}^{-2} \end{pmatrix}, \quad C := dg(\bar{U}) = \begin{pmatrix} 0 & 0 \\ -\bar{u}^2 & -2\bar{u}\bar{\tau} \end{pmatrix} \end{aligned}$$

are periodic functions of  $x$ . As the underlying solution  $\bar{U}$  depends on  $x$  only, (1.11) is clearly autonomous in time. By separation of variables, therefore, decomposing solutions into the sum of solutions of the form  $v(x, t) = e^{\lambda t}v(x)$ , where  $v$  satisfies the eigenvalue equation  $(L - \lambda)v = 0$ , or, equivalently, by taking the Laplace transform, we may reduce the study of stability of  $\bar{U}$  to the study of the spectral properties of the linearized operator  $L$ .

As the coefficients of  $L$  are  $X$ -periodic, Floquet theory implies that its spectrum is purely continuous. Moreover, its spectral properties may be conveniently analyzed by Bloch decomposition, an analogue for periodic-coefficient operators of the Fourier decomposition of a constant-coefficient operator, as we now describe.

**1.1.2. Bloch decomposition and stability conditions.** Following [5, 26, 27, 28], we define the family of operators

$$(1.13) \quad L_\xi = e^{-i\xi x} L e^{i\xi x} = (\partial_x + i\xi)B(\partial_x + i\xi) - (\partial_x + i\xi)A + C$$

operating on the class of  $L^2$  periodic functions on  $[0, X]$ ; the ( $L^2$ ) spectrum of  $L$  is equal to the union of the spectra of all  $L_\xi$  with  $\xi$  real with associated eigenfunctions

$$(1.14) \quad w(x, \xi, \lambda) := e^{i\xi x} q(x, \xi, \lambda),$$

where  $q$ , periodic, is an eigenfunction of  $L_\xi$ . By standard considerations [18],<sup>1</sup> the spectra of  $L_\xi$  consist of the union of countably many continuous surfaces  $\lambda_j(\xi)$ .

Without loss of generality, taking  $X = 1$ , recall now the *Bloch representation*

$$(1.15) \quad u(x) = \left(\frac{1}{2\pi}\right) \int_{-\pi}^{\pi} e^{i\xi \cdot x} \hat{u}(\xi, x) d\xi$$

of an  $L^2$  function  $u$ , where  $\hat{u}(\xi, x) := \sum_k e^{2\pi i k x} \hat{u}(\xi + 2\pi k)$  are periodic functions of period  $X = 1$ ,  $\hat{u}(\cdot)$  denoting with slight abuse of notation the Fourier transform of  $u$  in  $x$ . By Parseval's identity, the Bloch transform  $u(x) \rightarrow \hat{u}(\xi, x)$  is an isometry in  $L^2$ :

$$(1.16) \quad \|u\|_{L^2(x)} = \|\hat{u}\|_{L^2(\xi; L^2(x))},$$

where  $L^2(x)$  is taken on  $[0, 1]$  and  $L^2(\xi)$  on  $[-\pi, \pi]$ . Moreover, it diagonalizes the periodic-coefficient operator  $L$ , yielding the *inverse Bloch transform representation*

$$(1.17) \quad e^{Lt} u_0 = \left(\frac{1}{2\pi}\right) \int_{-\pi}^{\pi} e^{i\xi \cdot x} e^{L_\xi t} \hat{u}_0(\xi, x) d\xi,$$

relating behavior of the linearized system to that of the diagonal operators  $L_\xi$ .

Following [9], we assume along with (H1)–(H2) the following *strong spectral stability* conditions:

$$(D1) \quad \sigma(L_\xi) \subset \{\text{Re}\lambda < 0\} \text{ for } \xi \neq 0.$$

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<sup>1</sup>For example, the characterization [5] of spectra as the zero set of an associated Evans function.

(D2)  $\text{Re}\sigma(L_\xi) \leq -\theta|\xi|^2$ ,  $\theta > 0$ , for  $\xi \in \mathbb{R}$  and  $|\xi|$  sufficiently small.

(D3')  $\lambda = 0$  is an eigenvalue of  $L_0$  of multiplicity 2.<sup>2</sup>

As shown in [18], (H1)–(H2) and (D1)–(D3') imply that there exist 2 smooth eigenvalues

$$(1.18) \quad \lambda_j(\xi) = -ia_j\xi + o(|\xi|)$$

of  $L_\xi$  bifurcating from  $\lambda = 0$  at  $\xi = 0$ ; see Lemma 2.1 below.

Loosely following [9], we make the further nondegeneracy hypotheses:

(H3) The coefficients  $a_j$  in (1.18) are distinct.

(H4) The eigenvalue 0 of  $L_0$  is nonsemisimple, i.e.,  $\dim \ker L_0 = 1$ .

The coefficients  $a_j$  may be seen to be the characteristics of an associated Whitham averaged system,

$$(1.19) \quad \begin{aligned} M(\beta)_t + G(\beta)_x &= 0, \\ \Omega(\beta)_t + (c(\beta)\Omega(\beta))_x &= 0, \end{aligned}$$

linearized about the values of  $M$ ,  $G$ ,  $c$ ,  $\Omega$  associated with the background wave  $\bar{u}$ , where  $M$  is the mean of  $\tau$  over one period and  $F$  is the mean in the  $\tau$ -coordinate of a certain associated flux,  $c$  is the wave speed, and  $\Omega$  is the frequency of nearby periodic solutions, indexed as in (1.7) by  $\beta \in \mathbb{R}^2$ ; see [18, 22, 23].<sup>3</sup> System (1.19) formally governs slowly modulated solutions,

$$(1.20) \quad \tilde{u}(x, t) = \bar{u}^{\beta(\varepsilon x, \varepsilon t)}(\Psi(x, t)) + O(\varepsilon), \quad \varepsilon \rightarrow 0,$$

presumed to describe large spatiotemporal behavior  $x, t \gg 1$ , where  $\bar{u}^\beta(\cdot)$  as in (1.7) parametrizes the set of nearby periodic solutions,  $\Omega = \Psi_x$ , and  $c = -\Psi_t/\Psi_x$ .

Thus, (D1) implies weak hyperbolicity of the Whitham averaged system (1.19) (reality of  $a_j$ ), while (H3) corresponds to strict hyperbolicity. Condition (H4) holds generically and corresponds to the assumption that speed  $c$  is nonstationary along the manifold of nearby stationary solutions; see Lemma 2.1.<sup>4</sup> Condition (D2) corresponds to “diffusivity” of the large-time ( $\sim$  small frequency) behavior of the linearized system and holds generically given (H1)–(H4), (D1), and (D3').<sup>5</sup> Condition (D3') also holds generically and can be verified by an Evans function computation as described in [17]. As discussed in [20, 29, 8, 9], conditions (D1)–(D3') are conservation law analogues of the spectral assumptions introduced by Schneider in the reaction-diffusion case [26, 27, 28].

**1.2. Main result.**

**THEOREM 1.1.** *Assuming (H1)–(H4) and (D1)–(D3'), let  $\bar{U} = (\bar{\tau}, \bar{u})$  be a traveling-wave solution (1.3) of (1.2) satisfying the derivative condition*

$$(1.21) \quad 2\nu\bar{u}_x < F^{-1}.$$

<sup>2</sup>The zero eigenspace of  $L_0$ , corresponding to variations along the three-dimensional manifold of periodic solutions in directions for which period does not change [29, 9], is at least two-dimensional by linearized existence theory and (H2).

<sup>3</sup>Here, we follow the formalism and notation of [22, 23].

<sup>4</sup>The case that (H4) is violated may be treated as in [8].

<sup>5</sup>This amounts to nonvanishing of  $b_j$  in the Taylor series expansion  $\lambda_j(\xi) = -ia_j\xi - b_j\xi^2$  guaranteed by Lemma 2.1 given (H1)–(H4), (D1), and (D3').

Then, for some  $C > 0$  and  $\psi \in W^{K,\infty}(x, t)$ , where  $K \geq 3$  is as in (H1),

$$\begin{aligned}
 & \|\tilde{U} - \bar{U}(\cdot - \psi - ct)\|_{L^p}(t) \leq C(1+t)^{-\frac{1}{2}(1-1/p)} \|\tilde{U} - \bar{U}\|_{L^1 \cap H^K}|_{t=0}, \\
 (1.22) \quad & \|\tilde{U} - \bar{U}(\cdot - \psi - ct)\|_{H^K}(t) \leq C(1+t)^{-\frac{1}{4}} \|\tilde{U} - \bar{U}\|_{L^1 \cap H^K}|_{t=0}, \\
 & \|(\psi_t, \psi_x)\|_{W^{K+1,p}} \leq C(1+t)^{-\frac{1}{2}(1-1/p)} \|\tilde{U} - \bar{U}\|_{L^1 \cap H^K}|_{t=0},
 \end{aligned}$$

and

$$(1.23) \quad \|\tilde{U} - \bar{U}(\cdot - ct)\|_{L^\infty}(t), \|\psi(t)\|_{L^\infty} \leq C \|\tilde{U} - \bar{U}\|_{L^1 \cap H^K}|_{t=0}$$

for all  $t \geq 0$ ,  $p \geq 2$ , for solutions  $\tilde{U}$  of (1.2) with  $\|\tilde{U} - \bar{U}\|_{L^1 \cap H^K}|_{t=0}$  sufficiently small. In particular,  $\bar{U}$  is nonlinearly bounded  $L^1 \cap H^K \rightarrow L^\infty$  stable.

Theorem 1.1 asserts not only bounded  $L^1 \cap H^K \rightarrow L^\infty$  stability, a very weak notion of stability, but also asymptotic convergence of  $\tilde{U}$  to the modulated wave  $\bar{U}(x - \psi(x, t))$ .

*Remark 4.* With further effort, it may be shown that the results of Theorem 1.1 extend to all  $1 \leq p \leq \infty$  using the pointwise techniques of [21]; see [8, 9].

*Remark 5.* The derivative condition (1.21) is effectively an upper bound on the amplitude of the periodic wave; see Remark 9. As discussed in Remark 10, this is precisely the condition that the first-order part of the linearized equations (1.11) be symmetric hyperbolic (i.e., that  $A$  in (1.12) be symmetrizable) and reflects a subtle competition between hyperbolic and parabolic effects. (The first-order part of the inviscid equations is always symmetric-hyperbolic, corresponding to the equations of isentropic gas dynamics with  $\gamma$ -law gas.) It is satisfied when either wave amplitude or viscosity coefficient  $\nu$  is sufficiently small. It is not clear whether this condition may be relaxed.

We note that condition (1.21) is satisfied for *all* roll waves computed numerically in [18]. In the Eulerian coordinates considered in [18], (1.21) translates to  $h_x/h^3 < (2c_{Lagrangian}\nu F)^{-1}$ . Examining Figure 1 of [18], a phase portrait in  $(h, h')$  for  $F = 6$ ,  $\nu = 0.1$ , and  $c_{Lagrangian} \equiv 1$ ,<sup>6</sup> we see that all periodic orbits appear to satisfy  $h' \leq 0.5$ ,  $h \geq 1.3$  (worst case at bounding homoclinic), so that  $h'/h^3 \lesssim 0.228$ , whereas  $(2c_{Lagrangian}\nu F)^{-1} \gtrsim 0.8333$ . In Lagrangian coordinates, (1.21) is equivalent to the easier-to-verify condition

$$(1.24) \quad \bar{\tau}_x < (2c\nu F)^{-1},$$

which is readily checked within the phase portrait  $(\tau, \tau_x)$  of traveling-wave ODE (1.5).

It is straightforward using the bounds of Corollary 3.4 to show for “zero-mass,” or derivative, initial perturbations that nonlinear decay rates (1.22)–(1.23) improve by a factor of  $(1+t)^{-1/2}$  to the rates seen in the reaction-diffusion case [26, 10] for general (undifferentiated) localized perturbations. In particular, the perturbed wave  $\tilde{U}$  then decays asymptotically in  $L^\infty$  to the background wave  $\bar{U}$  with Gaussian rate  $(1+t)^{-1/2}$  as in the reaction-diffusion case. Likewise, under an unlocalized initial perturbation, or, equivalently, the integral of a localized perturbation, the difference between  $\tilde{U}$  and  $\bar{U}$  may be expected to blow up at rate  $(1+t)^{1/2}$ —this is indeed the linearized behavior—and, barring special nonlinear structure, there seems no reason

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<sup>6</sup>Here, we are using  $c_{Lagrangian} = (\text{mass flux through } (x - c_{Eulerian}t)) = h(c_{Eulerian} - u)$  together with the parametrization  $\bar{q} := h(c_{Eulerian} - u) \equiv 1$  of [18].

why the difference between  $\tilde{U}$  and the modulation  $\bar{U}(\cdot - \Psi)$  should not blow up as well: at best it remains bounded. In the reaction-diffusion case, for comparison, results announced in [25] assert that  $\tilde{U}$  remains close to  $\bar{U}$  even under unlocalized perturbations and approaches the modulated wave at rate  $(1+t)^{-1/2}$  in  $L^\infty$ . That is, the behavior in the conservation (balance) law case compared to that in the reaction-diffusion case is, roughly speaking, shifted by one derivative.<sup>7</sup>

This reflects a fundamental difference between modulational behavior in the present conservation (or balance) law setting from that of the reaction-diffusion case. Namely, in the reaction-diffusion case, the Whitham averaged system reduces to a single equation,  $\partial_t(\Omega) + \partial_x(\Omega c) = 0$ , or, equivalently,

$$(1.25) \quad \Psi_t + c(\Psi_x)\Psi_x = 0,$$

where  $\Omega := \Psi_x$  denotes frequency and  $c := -\frac{\Psi_t}{\Psi_x}$  the wave speed, and  $c$  and  $\Omega$  are related by the linearized dispersion relation along the family of periodic orbits (in the case considered by Schneider [26],  $c \equiv 0$ ). On the other hand, the Whitham averaged equations (1.19) in the present case are a genuine  $2 \times 2$  first-order hyperbolic system<sup>8</sup> in  $\Psi_x$  and wave speed  $c$ ,  $c$  now considered as an independent parameter; that is, they describe modulation of the perturbed wave in frequency  $\Psi_x$  and speed  $c$ , with phase shift  $\Psi$  determined indirectly by integration of  $\Psi_x$ .

Assuming heuristically (as justified at the linearized, spectral level by the Bloch analysis of section 2) that modulational behavior is governed by a second-order regularization of the first-order Whitham averaged system, we have the standard picture of behavior under localized perturbation as consisting of modulations in  $(\Psi_x, c)$  given by a pair of approximate Gaussians propagating outward with Whitham characteristic speeds  $a_1$  and  $a_2$ , hence an associated, much larger modulation in  $\Psi$  determined by integration in the  $\Psi_x$  component, given by a sum of approximate error functions propagating with the same speeds.

Indeed, this is exactly the description given in (3.27) of the principal part of the kernel  $e(x, t; y)$  determining  $\Psi$  through (4.23). Likewise, the principal part of the Green function of the linearized equations about  $\bar{U}$  is  $\bar{U}'(x)e(x, t; y)$ , showing that linearized behavior to lowest order indeed consists of a translation, or multiple of  $\bar{U}'(x)$ , with amplitude

$$\Psi(x, t) = \int e(x, t; y)(\tilde{U}(y, 0) - \bar{U}(y, 0))dy;$$

see the description of the Green function in Corollary 3.4.

This picture of modulational behavior as “filtering” by integration along a certain direction of the hyperbolic-parabolic system derived by Whitham averaging seems quite interesting at a phenomenological level and a genuinely novel aspect of the conservation (balance) law case. In particular, the  $\Psi_x$  component direction along which the integration is performed is in general independent of either characteristic mode, so that the resulting behavior is essentially different from that exemplified by (1.25) of a single scalar equation as in the reaction-diffusion case.

<sup>7</sup>At a purely technical level, this can be seen by the appearance of a Jordan block in the zero eigenspace of  $L_0$ , introducing factor  $\xi^{-1}$  in the description of low-frequency behavior (Lemma 2.1). Recall that a factor  $i\xi$  corresponds roughly to differentiation in the Bloch representation, through its relation to the Fourier transform. In the reaction-diffusion case, the zero eigenspace of  $L_0$  is simple, and no such factor appears.

<sup>8</sup>In general, the dimension of the Whitham averaged system is equal to the dimension of the manifold of nearby periodic solutions, modulo translations [9].

**1.3. Discussion and open problems.** The extension from the parabolic conservation law to the present case involves a number of new technical issues associated with lack of parabolicity and nonconservative form. We overcome these difficulties by combining the arguments of [8, 9, 18] with those of [16, 32, 30] (real viscosity) and [13] and [12, 31] (relaxation and combustion systems, both involving nonconservative terms).

An interesting open problem is the rigorous justification of spectral stability of roll waves approaching the inviscid case in the singular zero viscosity limit, extending results of [18]. We hope to carry this out in future work. For related asymptotic analysis, see the study in [33] of the inviscid limit for detonations.

Another interesting open problem is the numerical investigation of spectral stability of large-amplitude roll waves. In particular, it is an interesting question whether violation of the apparently technical “amplitude condition” (1.21) corresponds to actual physical phenomena/instability. This is not inconceivable, as (1.21) is needed in our argument not only for nonlinear iteration, but also for high-frequency linearized bounds. As this is the condition that the first-order part of the equations be symmetric hyperbolic, it may well have such significance—however, this is not yet clear.

It is straightforward to extend our results to the two-dimensional small-amplitude case by working in Eulerian coordinates and substituting for the present large-amplitude damping estimate the simpler small-amplitude version of [14]; see [8, 9] for the multidimensional analysis of periodic waves. However, there is some evidence that roll waves develop transverse instabilities in multidimensions [19]. If so, this suggests the question of whether such instability might be connected with bifurcation to multiply periodic waves. The extension of our stability analysis to the multiply periodic case, as suggested in [8, 9], is another very interesting open problem.

**2. Spectral preparation.** We begin by a careful study of the Bloch perturbation expansion near  $\xi = 0$ .

**LEMMA 2.1.** *Assuming (H1)–(H4), (D1), and (D3'), the eigenvalues  $\lambda_j(\xi)$  of  $L_\xi$  are analytic functions and the Jordan structure of the zero eigenspace of  $L_0$  consists of a one-dimensional kernel and a single Jordan chain of height 2, where the left kernel of  $L_0$  is spanned by the constant function  $\tilde{f} \equiv (1, 0)^T$ , and  $\tilde{u}'$  spans the right eigendirection lying at the base of the Jordan chain. Moreover, for  $|\xi|$  sufficiently small, there exist right and left eigenfunctions  $q_j(\xi, \cdot)$  and  $\tilde{q}_j(\xi, \cdot)$  of  $L_\xi$  associated with  $\lambda_j$  of the forms  $q_j = \sum_{k=1}^2 \beta_{j,k} v_k$  and  $\tilde{q}_j = \sum_{k=1}^2 \tilde{\beta}_{j,k} \tilde{v}_k$ , where  $\{v_j\}_{j=1}^2$  and  $\{\tilde{v}_j\}_{j=1}^2$  are dual bases of the total eigenspace of  $L_\xi$  associated with sufficiently small eigenvalues, analytic in  $\xi$ , with  $\tilde{v}_2(0)$  constant and  $v_1(0) \equiv \tilde{u}'(\cdot)$ ;  $\xi^{-1} \tilde{\beta}_{j,1}, \tilde{\beta}_{j,2}$  and  $\xi \beta_{j,1}, \beta_{j,2}$  are analytic in  $\xi$ ; and  $\langle \tilde{q}_j, q_k \rangle = \delta_j^k$ .*

*Remark 6.* Notice that the results of Lemma 2.1 are somewhat unexpected since, in general, eigenvalues bifurcating from a nontrivial Jordan block typically do so in a nonanalytic fashion, rather being expressed in a Puiseux series in fractional powers of  $\xi$ .<sup>9</sup> The fact that analyticity prevails in our situation is a consequence of the very special structure of the left and right generalized null-spaces of the unperturbed operator  $L_0$ , and the special forms of the equations considered.

<sup>9</sup>This is, however, consistent with the picture of behavior as being approximately governed by a first-order Whitham averaged system with eigenvalue perturbation expansions agreeing to first-order with the associated linearized homogeneous dispersion relation [18, 22, 23].



*Proof.* Recall that  $L_\xi$  has a spectrum consisting of isolated eigenvalues of finite multiplicity [18, 5]. Expanding

$$(2.1) \quad L_\xi = L_0 + i\xi L^1 - \xi^2 L^2,$$

where, by (1.13),

$$(2.2) \quad L_0 = \partial_x B \partial_x - \partial_x A + C, \quad L^1 = (B \partial_x + \partial_x B - A), \quad L^2 = B,$$

consider the spectral perturbation problem in  $\xi$  about the eigenvalue  $\lambda = 0$  of  $L_0$ .

Because 0 is an isolated eigenvalue of  $L_0$ , the associated total right and left eigenprojections  $P_0$  and  $\tilde{P}_0$  perturb analytically in  $\xi$ , giving projection  $P_\xi$  and  $\tilde{P}_\xi$  [11]. These yield in standard fashion (for example, by projecting appropriately chosen fixed subspaces) locally analytic right and left bases  $\{v_j\}$  and  $\{\tilde{v}_j\}$  of the associated total eigenspaces given by the range of  $P_\xi, \tilde{P}_\xi$ .

Defining  $V = (v_1, v_2)$  and  $\tilde{V} = (\tilde{v}_1, \tilde{v}_2)^*$ ,  $*$  denoting adjoint, we may convert the infinite-dimensional perturbation problem (2.1) into a  $2 \times 2$  matrix perturbation problem

$$(2.3) \quad M_\xi = M_0 + i\xi M_1 - \xi^2 M_2 + O(|\xi|^3),$$

where  $M_\xi := \langle \tilde{V}_\xi^*, L_\xi V_\xi \rangle$  and  $\langle \cdot, \cdot \rangle$  denotes the standard  $L^2(x)$  inner product on the finite interval  $[0, X]$ . That is, the eigenvalues  $\lambda_j(\xi)$  lying near 0 of  $L_\xi$  are the eigenvalues of  $M_\xi$ , and the associated right and left eigenfunctions of  $L_\xi$  are

$$(2.4) \quad f_j = V w_j \quad \text{and} \quad \tilde{f}_j = \tilde{w}_j \tilde{V}^*,$$

where  $w_j$  and  $\tilde{w}_j$  are the associated right and left eigenvectors of  $M_\xi$ .

By assumption,  $\lambda = 0$  is a nonsemisimple eigenvalue of  $L_0$ , so that  $M_0$  is nilpotent but nonzero, possessing a nontrivial associated Jordan chain. Moreover, using the fact that  $\langle (1, 0)^T, C \rangle = 0$ , where, again,  $\langle \cdot, \cdot \rangle$  represents the  $L^2(x)$  inner product over the finite domain  $[0, X]$ , the function  $\tilde{f} \equiv (1, 0)^T$  by direct computation lies in the kernel of  $L_0^* = (\partial_x B^* \partial_x + A^* \partial_x + C^*)$ , and we have that the two-dimensional zero eigenspace of  $L_0$  consists precisely of a one-dimensional kernel and a single Jordan chain of height two. Moreover, by translation-invariance (differentiate in  $x$  the profile equation (1.5)), we have  $L_0 \bar{u}' = 0$ , so that  $\bar{u}'$  lies in the right kernel of  $L_0$ .

Now, recall by assumption (H2) that  $H : \mathbb{R}^5 \rightarrow \mathbb{R}^2$ , taking  $(X, c, q, b) \mapsto (\tau, \tau')(X, c, b; X) - b$  is full rank at  $(\bar{X}, \bar{c}, \bar{b})$ , where  $(\tau, \tau')(\cdot; \cdot)$  is the solution operator of (1.5). The fact that  $\ker L_0$  is one-dimensional implies that the restriction  $\tilde{H}$  taking  $(b, q) \mapsto u(X; b, c, q) - b$  for fixed  $(X, c)$  is also full rank; i.e.,  $H$  is full rank with respect to the specific parameters  $(X, c)$ . Applying the implicit function theorem and counting dimensions, we find that the set of periodic solutions, i.e., the inverse image of zero under map  $H$  local to  $\bar{u}$ , is a smooth three-dimensional manifold  $\{\bar{u}^\beta(x - \alpha - c(\beta)t)\}$ , with  $\alpha \in \mathbb{R}, \beta \in \mathbb{R}^2$ . Moreover, two dimensions may be parametrized by  $(X, c)$ , or, without loss of generality,  $\beta = (X, c)$ .

Fixing  $X$  and varying  $c$ , we find by differentiation of (1.5) that  $f_* := -\partial_s \bar{U}$  satisfies the generalized eigenfunction equation

$$L_0 f_* = \bar{U}'.$$

Thus,  $\bar{U}'$  spans the eigendirection lying at the base of the Jordan chain, with the generalized zero-eigenfunction of  $L_0$  corresponding to variations in speed along the

manifold of periodic solutions about  $\bar{U}$ . Without loss of generality, therefore, we may take  $\tilde{v}_2$  to be constant at  $\xi = 0$ , and  $v_1 \equiv \bar{U}'$  at  $\xi = 0$ .

Noting as in [8] the fact that, by (1.12),

$$(2.5) \quad \begin{aligned} A\bar{U}_x &= f(\bar{u})_x - (\partial_x B(\bar{u}))\bar{U}_x = \partial_x(f(\bar{u})_x - B(\bar{U})\bar{U}_x) + B(\bar{u})\partial_x \bar{u}_x \\ &= g(\bar{U}) + B(\bar{u})\partial_x \bar{u}_x, \end{aligned}$$

and so by  $e_2 g = 0, \partial_x e_2 = 0$ , we have

$$\langle e_2, L^1 \bar{U}' \rangle = \langle e_2, (\partial_x B + B\partial_x - A)\bar{U}' \rangle = \langle e_2, \partial_x B \bar{U}' \rangle \equiv 0$$

for  $e_2 := (0, 1)$ , where  $\langle \cdot, \cdot \rangle$  denotes the  $L^2(x)$  inner product on the interval  $x \in [0, X]$ , we find under this normalization that (2.3) has the special structure

$$(2.6) \quad M_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

Now, rescaling (2.3) as

$$(2.7) \quad \check{M}_\xi := (i\xi)^{-1} S(\xi) M_\xi S(\xi)^{-1},$$

where

$$(2.8) \quad S := \begin{pmatrix} i\xi & 0 \\ 0 & 1 \end{pmatrix},$$

we obtain

$$(2.9) \quad \check{M}_\xi = \check{M}_0 + i\xi \check{M}_1 + O(\xi^2),$$

where  $\check{M}_j$  like the original  $M_j$  are constant and the eigenvalues  $m_j(\xi)$  of  $\check{M}_\xi$  are  $(i\xi)^{-1} \lambda_j(\xi)$ .

As the eigenvalues  $m_j$  of  $\check{M}_\xi$  are continuous, the eigenvalues  $\lambda_j(\xi) = i\xi m_j$  are differentiable at  $\xi = 0$  as asserted in the introduction. Moreover, by (H3), the eigenvalues  $\check{\lambda}_j(0)$  of  $\check{M}_0$  are distinct, and so they perturb analytically in  $\xi$ , as do the associated right and left eigenvectors  $z_j$  and  $\tilde{z}_j$ . Undoing the rescaling (2.7) and recalling (2.4), we obtain the result.  $\square$

**3. Linearized stability estimates.** By standard spectral perturbation theory [11], the total eigenprojection  $P(\xi)$  onto the eigenspace of  $L_\xi$  associated with the eigenvalues  $\lambda_j(\xi)$ ,  $j = 1, 2$ , described in the previous section is well defined and analytic in  $\xi$  for  $\xi$  sufficiently small, since these (by discreteness of the spectra of  $L_\xi$ ) are separated at  $\xi = 0$  from the rest of the spectrum of  $L_0$ . By (D2), there exists an  $\varepsilon > 0$  such that  $\Re \lambda_j(\xi) \leq -\theta |\xi|^2$  for  $0 < |\xi| < 2\varepsilon$ . With this choice of  $\varepsilon$ , we introduce a smooth cutoff function  $\phi(\xi)$  that is identically one for  $|\xi| \leq \varepsilon$  and identically zero for  $|\xi| \geq 2\varepsilon$ ,  $\varepsilon > 0$  sufficiently small; we split the solution operator  $S(t) := e^{Lt}$  into a low-frequency part,

$$(3.1) \quad S^I(t)u_0 := \left(\frac{1}{2\pi}\right) \int_{-\pi}^\pi e^{i\xi \cdot x} \phi(\xi) P(\xi) e^{L_\xi t} \hat{u}_0(\xi, x) d\xi,$$

and the associated high-frequency part,

$$(3.2) \quad S^{II}(t)U_0 := \left(\frac{1}{2\pi}\right) \int_{-\pi}^\pi e^{i\xi \cdot x} (I - \phi P(\xi)) e^{L_\xi t} \hat{U}_0(\xi, x) d\xi.$$

Our strategy is to treat the high- and low-frequency operators separately since, as is standard, the low-frequency analysis is considerably more complicated than the corresponding high-frequency analysis. That being said, we begin by deriving bounds on the solution operator at high frequency.

**3.1. High-frequency bounds.** By boundedness of the resolvent on compact subdomains of the resolvent set, equivalence (as the zero-set of an associated Evans function [18, 20]) of  $H^1$  and  $L^2$  spectrum, assumption (D2), and the high-frequency estimates of Lemma B.1, we have for  $|\xi|$  bounded away from zero and waves satisfying the amplitude condition (1.21) that the resolvent  $(\lambda - L_\xi)^{-1}$  is uniformly bounded from  $H^1 \rightarrow H^1$  for  $\Re \lambda = -\eta < \theta < 0$ , whence, by Prüss' theorem [24],  $\|e^{L_\xi t} f\|_{H^1} \leq C e^{-\theta t} \|f\|_{H^1}$ .

For  $|\xi|$  sufficiently small, on the other hand,  $\phi \equiv 1$ , and  $I - \phi(\xi)P = I - P = Q$ , where  $Q$  is the eigenprojection of  $L_\xi$  associated with eigenvalues complementary to  $\lambda_j(\xi)$ , which by spectral separation of  $\lambda_j(\xi)$  from the remaining spectra of  $L_\xi$  have real parts strictly less than zero. Applying Prüss' theorem to the restriction of  $L_\xi$  to the Hilbert space given by the range of  $Q$ , we find, likewise, that  $\|e^{L_\xi t}(I - \phi(\xi))f\|_{H^1} = \|e^{L_\xi t} Q f\|_{H^1} \leq C e^{-\theta t} \|f\|_{H^1}$ .

Combining these observations, we have the exponential decay bound

$$\|e^{L_\xi t}(I - \phi P(\xi))f\|_{H^1([0,X])} \leq C e^{-\theta t} \|f\|_{H^1([0,X])}$$

for  $\theta > 0$  as in (D2) and  $C > 0$ , from which it follows that

$$(3.3) \quad \|e^{L_\xi t}(I - \phi P(\xi))\partial_x^l f\|_{H^1([0,X])} \leq C e^{-\theta t} \|f\|_{H^{l+1}([0,X])}$$

for  $0 \leq l \leq K$  ( $K$  as in (H1)). Together with (1.16), these give immediately the following estimates.

PROPOSITION 3.1 (see [23]). *Under assumptions (H1)–(H4), (D1)–(D2), and assuming the amplitude condition (1.21) holds, there exist constants  $\theta, C > 0$ , such that for all  $t > 0$ ,  $2 \leq p \leq \infty$ ,  $0 \leq l \leq 2$ ,  $0 \leq m \leq 2$ , we have the high-frequency estimates*

$$(3.4) \quad \begin{aligned} \|S^{II}(t)\partial_x^l f\|_{L^2(x)} &\leq C e^{-\theta t} \|f\|_{H^{l+1}(x)}, \\ \|S^{II}(t)\partial_x^m f\|_{L^p(x)} &\leq C e^{-\theta t} \|f\|_{H^{m+2}(x)}. \end{aligned}$$

*Proof.* For  $m, l = 0$ , the first inequalities follow immediately by (1.16) and (3.3). The second follows for  $p = \infty$  by Sobolev embedding. The result for general  $2 \leq p \leq \infty$  then follows by  $L^p$  interpolation. A similar argument applies for  $1 \leq l, m \leq 2$  by higher-derivative versions of (3.3), which follow in exactly the same way.  $\square$

**3.2. Low-frequency bounds.** As noted above, analysis of the solution operator at low frequency is considerably more complicated than the high-frequency bounds outlined above. To aid in our analysis, we introduce the Green kernel

$$(3.5) \quad G^I(x, t; y) := S^I(t)\delta_y(x)$$

associated with  $S^I$ , and the corresponding kernel

$$(3.6) \quad [G_\xi^I(x, t; y)] := \phi(\xi)P(\xi)e^{L_\xi t}[\delta_y(x)]$$

appearing within the Bloch representation of  $G^I$ , where the brackets on  $[G_\xi]$  and  $[\delta_y]$  denote the periodic extensions of these functions onto the whole line. Then we have

the following descriptions of  $G^I$ ,  $[G_\xi^I]$ , deriving from the spectral expansion (1.18) of  $L_\xi$  near  $\xi = 0$ .

PROPOSITION 3.2 (see [23]). *Under assumptions (H1)–(H4) and (D1)–(D3'),*

$$\begin{aligned}
 [G_\xi^I(x, t; y)] &= \phi(\xi) \sum_{j=1}^2 e^{\lambda_j(\xi)t} q_j(\xi, x) \tilde{q}_j(\xi, y)^*, \\
 (3.7) \quad G^I(x, t; y) &= \left(\frac{1}{2\pi}\right) \int_{\mathbb{R}} e^{i\xi \cdot (x-y)} [G_\xi^I(x, t; y)] d\xi \\
 &= \left(\frac{1}{2\pi}\right) \int_{\mathbb{R}} e^{i\xi \cdot (x-y)} \phi(\xi) \sum_{j=1}^2 e^{\lambda_j(\xi)t} q_j(\xi, x) \tilde{q}_j(\xi, y)^* d\xi,
 \end{aligned}$$

where  $*$  denotes matrix adjoint, or complex conjugate transpose, and  $q_j(\xi, \cdot)$  and  $\tilde{q}_j(\xi, \cdot)$  are right and left eigenfunctions of  $L_\xi$  associated with eigenvalues  $\lambda_j(\xi)$  defined in (1.18), normalized so that  $\langle \tilde{q}_j, q_j \rangle \equiv 1$ .

*Proof.* Relation (3.7)<sub>1</sub> is immediate from the spectral decomposition for  $C^0$  semi-groups at eigenvalues of finite multiplicity, and the fact that  $\lambda_j$  are distinct for  $|\xi| > 0$  sufficiently small, by (H3). Substituting (3.5) into (3.1) and computing

$$(3.8) \quad \widehat{\delta}_y(\xi, x) = \sum_k e^{2\pi i k x} \widehat{\delta}_y(\xi + 2\pi k e_1) = \sum_k e^{2\pi i k x} e^{-i\xi \cdot y - 2\pi i k y} = e^{-i\xi \cdot y} [\delta_y(x)],$$

where the second and third equalities follow from the fact that the Fourier transform of either the continuous or discrete delta-function is unity, we obtain

$$\begin{aligned}
 G^I(x, t; y) &= \left(\frac{1}{2\pi}\right) \int_{-\pi}^{\pi} e^{i\xi \cdot x} \phi P(\xi) e^{L_\xi t} \widehat{\delta}_y(\xi, x) d\xi \\
 &= \left(\frac{1}{2\pi}\right) \int_{-\pi}^{\pi} e^{i\xi \cdot (x-y)} \phi P(\xi) e^{L_\xi t} [\delta_y(x)] d\xi,
 \end{aligned}$$

yielding (3.7)<sub>2</sub> by (3.6) and the fact that  $\phi$  is supported on  $[-\pi, \pi]$ . □

We now state our main result for this section, which uses the spectral representation of  $G^I$  and  $[G_\xi^I]$  described in Proposition 3.2 to decompose the low-frequency Green kernel into a leading order piece (corresponding to translational modulation) plus a faster decaying residual. Underlying this decomposition is the fundamental relation

$$(3.9) \quad G(x, t; y) = \left(\frac{1}{2\pi}\right) \int_{-\pi}^{\pi} \int_{\mathbb{R}^{d-1}} e^{i\xi \cdot (x-y)} [G_\xi(x_1, t; y_1)] d\xi,$$

which serves as the crux of the low-frequency analysis both here and in [21, 8].

PROPOSITION 3.3. *Under assumptions (H1)–(H4) and (D1)–(D3'), the low-frequency Green function  $G^I(x, t; y)$  of (3.5) decomposes as  $G^I = E + \tilde{G}^I$ ,*

$$(3.10) \quad E = \bar{U}'(x) e(x, t; y),$$

where, for some  $C > 0$  and all  $t > 0$ ,

$$\begin{aligned}
 (3.11) \quad \sup_y \|\tilde{G}^I(\cdot, t; y)\|_{L^p(x)} &\leq C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}, \\
 \sup_y \|\partial_t^r \tilde{G}^I(\cdot, t; y)\|_{L^p(x)}, \quad \sup_y \|\partial_t^l \tilde{G}^I(\cdot, t; y)\|_{L^p(x)} &\leq C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}, \\
 \sup_y \|\tilde{G}^I(\cdot, t; y)(0, 1)^T\|_{L^p(x)} &\leq C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}
 \end{aligned}$$

for  $p \geq 2, 1 \leq r \leq 2,$

$$(3.12) \quad \sup_y \|\partial_x^j \partial_t^l \partial_y^r e(\cdot, t, ; y)\|_{L^p(x)} \leq C(1+t)^{-\frac{1}{2}(1-\frac{1}{p}) - \frac{(j+l)}{2} - \frac{1}{2}}$$

for  $p \geq 2, 0 \leq j, l, j+l \leq K+1, 1 \leq r \leq 2,$  and

$$(3.13) \quad \sup_y \|\tilde{\partial}_x^j \partial_t^l e(\cdot, t, ; y)\|_{L^p(x)} \leq C(1+t)^{-\frac{1}{2}(1-\frac{1}{p}) - \frac{(j+l)}{2}}$$

for  $0 \leq j, l, j+l \leq K+1,$  provided that  $p \geq 2$  and  $j+l \geq 1$  or  $p = \infty.$  Moreover,  $e(x, t; y) \equiv 0$  for  $t \leq 1.$

*Remark 7.* The crucial new observation in the nonconservative case treated here is (3.11)<sub>3</sub>, which asserts that sources entering in the nonconservative second coordinate of the linearized equations experience decay equivalent to that of a differentiated source entering in the first coordinate. This is what allows us to treat nondivergence-form source terms arising in the second equation of the eventual perturbation equations.

*Proof.* Recalling (3.7) and Lemma 2.1, we have

$$(3.14) \quad \begin{aligned} G^I(x, t; y) &= \left(\frac{1}{2\pi}\right) \int_{\mathbb{R}} e^{i\xi \cdot (x-y)} \phi(\xi) \sum_{j=1}^2 e^{\lambda_j(\xi)t} q_j(\xi, x) \tilde{q}_j(\xi, y)^* d\xi \\ &= \left(\frac{1}{2\pi}\right) \int_{\mathbb{R}} e^{i\xi \cdot (x-y)} \phi(\xi) \sum_{j,k,l=1}^2 e^{\lambda_j(\xi)t} \beta_{j,k} v_k(\xi, x) \tilde{\beta}_{j,l} \tilde{v}_l(\xi, y)^* d\xi; \end{aligned}$$

the fact that  $\beta_{j,1} = O(\xi^{-1})$  suggests the  $k = 1$  terms (corresponding to translation) dominate the low-frequency Green kernel. With this motivation, we define

$$(3.15) \quad \tilde{e}(x, t; y) = \left(\frac{1}{2\pi}\right) \int_{\mathbb{R}} e^{i\xi \cdot (x-y)} \phi(\xi) \sum_{j,l} e^{\lambda_j(\xi)t} \beta_{j,1} \tilde{\beta}_{j,l} \tilde{v}_l(\xi, y)^* d\xi$$

so that

$$(3.16) \quad \begin{aligned} G^I(x, t; y) - \bar{U}'(x) \tilde{e}(x, t; y) &= \left(\frac{1}{2\pi}\right) \int_{\mathbb{R}} e^{i\xi \cdot (x-y)} \phi(\xi) \sum_{j,k \neq 1,l} e^{\lambda_j(\xi)t} \beta_{j,k} \tilde{\beta}_{j,l} v_k(\xi, x) \tilde{v}_l(\xi, y)^* d\xi \\ &\quad + \left(\frac{1}{2\pi}\right) \int_{\mathbb{R}} e^{i\xi \cdot (x-y)} \phi(\xi) \sum_{j,l} e^{\lambda_j(\xi)t} \beta_{j,1} \tilde{\beta}_{j,l} \left(v_1(\xi, x) - \bar{U}'(x)\right) \tilde{v}_l(\xi, y)^* d\xi, \end{aligned}$$

where, by analyticity of  $v_1, v_1(\xi, x) - \bar{U}'(x) = O(|\xi|),$  and so, by Lemma 2.1,

$$(3.17) \quad \beta_{j,1} \tilde{\beta}_{j,l} \left(v_1(\xi, x) - \bar{U}'(x)\right) \tilde{v}_l(\xi, y)^* = O(1)$$

and

$$(3.18) \quad \beta_{j,2} \tilde{\beta}_{j,l} v_2(\xi, x) \tilde{v}_l(\xi, y)^* = O(1).$$

Note further that  $\tilde{v}_l \equiv (1, 0)^T$  unless  $l = 1,$  in which case  $\tilde{\beta}_{j,l} = O(|\xi|)$  by Lemma 2.1; hence

$$(3.19) \quad \partial_y \left(\beta_{j,1} \tilde{\beta}_{j,l} \left(v_1(\xi, x) - \bar{U}'(x)\right) \tilde{v}_l(\xi, y)^*\right) = O(|\xi|),$$

$$(3.20) \quad \left(\beta_{j,1}\tilde{\beta}_{j,l}\left(v_1(\xi, x) - \bar{U}'(x)\right)\tilde{v}_l(\xi, y)^*\right)(0, 1)^T = O(|\xi|),$$

and

$$(3.21) \quad \partial_y\left(\beta_{j,2}\tilde{\beta}_{j,l}v_2(\xi, x)\tilde{v}_l(\xi, y)^*\right) = O(|\xi|),$$

$$(3.22) \quad \left(\beta_{j,2}\tilde{\beta}_{j,l}v_2(\xi, x)\tilde{v}_l(\xi, y)^*\right)(0, 1)^T = O(|\xi|).$$

From representation (3.16), bounds (3.17)–(3.18), and  $\Re\lambda_j(\xi) \leq -\theta|\xi|^2$ , we obtain by the triangle inequality

$$(3.23) \quad \|\tilde{G}^I(\cdot, t; \cdot)\|_{L^\infty(x,y)} = \|G^I - \bar{U}'\tilde{e}\|_{L^\infty(x,y)} \leq C\|e^{-\theta|\xi|^2t}\phi(\xi)\|_{L^1(\xi)} \leq C(1+t)^{-\frac{1}{2}}.$$

Derivative bounds follow similarly, since  $x$ -derivatives falling on  $v_{jk}$  are harmless, whereas, by (3.19)–(3.21),  $y$ - or  $t$ -derivatives falling on  $\tilde{v}_{jl}$  or on  $e^{i\xi\cdot(x-y)}$  introduce a factor of  $|\xi|$ , improving the decay rate by a factor of  $(1+t)^{-1/2}$ . (Note that  $|\xi|$  is bounded because of the cutoff function  $\phi$ , so there is no singularity at  $t = 0$ .)

To obtain the corresponding bounds for  $p = 2$ , we note that (3.14) may be viewed itself as a Bloch decomposition with respect to variable  $z := x - y$ , with  $y$  appearing as a parameter. Recalling (1.16), we may thus estimate

$$(3.24) \quad \begin{aligned} &\sup_y \|G^I(\cdot, t; y) - \bar{U}'\tilde{e}(\cdot, t; y)\|_{L^2(x)} \\ &\leq C \sum_{j,k \neq 1,l} \sup_y \|\phi(\xi)e^{\lambda_j(\xi)t}v_k(\cdot, z_1)\tilde{v}_l^*(\cdot, y)\tilde{v}_l(\cdot, y)^*\|_{L^2(\xi; L^2(z_1 \in [0, X]))} \\ &\quad + C \sum_{j,l} \sup_y \left\| \phi(\xi)e^{\lambda_j(\xi)t} \left( \frac{v_n(\cdot, x) - \bar{U}'(x)}{|\cdot|} \right) \tilde{v}_l(\cdot, y)^* \right\|_{L^2(\xi; L^2(z_1 \in [0, X]))} \\ &\leq C \sum_{j,k \neq 1,l} \sup_y \|\phi(\xi)e^{-\theta|\xi|^2t}\|_{L^2(\xi)} \sup_\xi \|v_k(\cdot, z_1)\|_{L^2(0, X)} \|\tilde{v}_l(\cdot, y)^*\|_{L^\infty(0, X)} \\ &\quad + C \sum_{j,l} \sup_y \|\phi(\xi)e^{-\theta|\xi|^2t}\|_{L^2(\xi)} \sup_\xi \left\| \left( \frac{v_n(\xi, x) - \bar{U}'(x)}{|\xi|} \right) \right\|_{L^2(0, X)} \|\tilde{v}_l(\cdot, y)^*\|_{L^\infty(0, X)} \\ &\leq C(1+t)^{-\frac{1}{4}}, \end{aligned}$$

where we have used in a crucial way the boundedness of  $\tilde{v}_l$  in  $L^\infty$ ,<sup>10</sup> and also the boundedness of

$$\left(\frac{v_n(\xi, x) - \bar{U}'(x)}{\xi}\right) \sim \partial_\xi v_n(r)$$

in  $L^2$ , where  $r \in (0, \xi)$ . Derivative bounds follow similarly as above, noting that  $y$ - or  $t$ -derivatives introduce a factor  $\xi$ , while  $x$ -derivatives are harmless, to obtain an

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<sup>10</sup>This is clear for  $\xi = 0$ , since  $v_j$  are linear combinations of genuine and generalized eigenfunctions, which are solutions of the homogeneous or inhomogeneous eigenvalue ODE. More generally, note that the resolvent of  $L_\xi - \gamma$  gains one derivative, and hence the total eigenprojection, as a contour integral of the resolvent, does too; now, use the one-dimensional Sobolev inequality for periodic boundary conditions to bound the  $L^\infty$  difference from the mean by the (bounded)  $H^1$  norm, then bound the mean by the  $L^1$  norm, which is controlled by the  $L^2$  norm.

additional factor of  $(1 + t)^{-1/2}$  decay. Finally, bounds for  $2 \leq p \leq \infty$  follow by  $L^p$  interpolation.

Now, defining

$$(3.25) \quad e(x, t; y) := \chi(t)\tilde{e}(x, t; y),$$

where  $\tilde{e}$  is defined in (3.15) and  $\chi$  is a smooth cutoff function such that  $\chi(t) \equiv 1$  for  $t \geq 2$  and  $\chi(t) \equiv 0$  for  $t \leq 1$ , and setting  $\tilde{G} := G - \tilde{U}'(x)e(x, t; y)$ , we readily obtain the estimates (3.11) by combining the above estimates on  $G^I - \tilde{U}\tilde{e}$  with bound (3.4) on  $G^{II}$ .

Finally, recalling, by Lemma 2.1, that  $\tilde{v}_l \equiv \text{const}$  for  $l \neq 1$  while  $\tilde{\beta}_{j,1} = O(|\xi|)$ , we have

$$\partial_y \left( \beta_{j,1} \tilde{\beta}_{j,l} \tilde{v}_l(\xi, y)^* \right) = o(|\xi|).$$

Bounds (3.12) thus follow from (3.15) by the argument used to prove (3.11), together with the observation that  $x$ - or  $t$ -derivatives introduce factors of  $\xi$ . Bounds (3.13) follow similarly for  $j + l \geq 1$ , in which case the integrand on the right-hand side of (3.15) (now differentiated in  $x$  and/or  $t$ ) is Lebesgue integrable.

In the critical case  $j = l = 0$ , taking  $t$  without loss of generality  $\geq 1$ , expanding

$$\lambda_j(\xi) = -i\xi a_j - b_j \xi^2 + O(\xi^3),$$

and setting  $\check{\lambda}(\xi) := -i\xi a_j - b_j \xi^2$ , we may write  $\tilde{e}(x, t; y)$  in (3.15) as

$$(3.26) \quad \begin{aligned} & \left( \frac{1}{2\pi} \right) \int_{\mathbb{R}} \sum_j \check{\beta}_{j,1}(0) \check{\beta}_{j,2}(0) \tilde{v}_2(0, y)^* e^{i\xi \cdot (x-y)} \xi^{-1} e^{\check{\lambda}_j(\xi)t} d\xi \\ &= \left( \frac{1}{2\pi} \right) \text{P.V.} \int_{\mathbb{R}} \sum_j \check{\beta}_{j,1}(0) \check{\beta}_{j,2}(0) \tilde{v}_2(0, y)^* e^{i\xi \cdot (x-y)} \xi^{-1} e^{\check{\lambda}_j(\xi)t} d\xi \\ &= \sum_j \check{\beta}_{j,1}(0) \check{\beta}_{j,2}(0) \tilde{v}_2(0, y)^* \left( \frac{1}{2\pi} \right) \text{P.V.} \int_{\mathbb{R}} e^{i\xi \cdot (x-y)} \xi^{-1} e^{\check{\lambda}_j(\xi)t} d\xi, \end{aligned}$$

where  $\check{\beta}_{j,1}(0) := \lim_{\xi \rightarrow 0} (\xi \beta_{j,1}(\xi))$ , and the above series is convergent by the alternating series test, plus a negligible error term

$$\left( \frac{1}{2\pi} \right) \text{P.V.} \int_{\mathbb{R}} e^{i\xi \cdot (x-y)} \phi(\xi) O(e^{-\theta|\xi|^2 t}) d\xi$$

for which the integrand is Lebesgue integrable and hence, by the previous argument, obeys the bounds for  $j + l = 1$ . (Note that the integral on the left-hand side of (3.26) is absolutely convergent by  $\xi^{-1}(e^{-ia_1 \xi t} - e^{-ia_2 \xi t}) \sim |a_1 - a_2|t$ , becoming conditionally convergent only when the integrand is split into different eigenmodes.)

By (D2), we have  $a_j$  real and  $\Re b_j > 0$ . Moreover, the operator  $L$ , since it is real-valued, has a spectrum with complex conjugate symmetry; hence  $b_j$  is real as well. Observing that  $\left( \frac{1}{2\pi} \right) \text{P.V.} \int_{\mathbb{R}} e^{i\xi \cdot (x-y)} \xi^{-1} e^{\check{\lambda}_j(\xi)t} d\xi$  is an antiderivative in  $x$  of the inverse Fourier transform  $\left( \frac{1}{2\pi} \right) \int_{\mathbb{R}} e^{i\xi \cdot (x-y)} e^{\check{\lambda}_j(\xi)t} d\xi = \frac{e^{-(x-y-a_j t)^2 / 4b_j t}}{\sqrt{4\pi b_j t}}$ , a Gaussian, we find that the principal part (3.26) is a sum of error functions

$$(3.27) \quad \sum_{j=1}^2 c_j \text{erf}\left( \frac{x - y - a_j t}{\sqrt{4b_j t}}, t \right) \tilde{v}_2(0, y),$$

hence bounded in  $L^\infty$  as claimed, where  $a_j$  denotes the characteristic speeds of the Whitham averaged system and (on further inspection)  $\sum_j c_j = 0$ . This verifies bound (3.13) in the final case  $j = l = 0$ , completing the proof.  $\square$

*Remark 8.* See [21, proof of Proposition 1.5] for an essentially equivalent estimate from the inverse Laplace transform point of view of the critical  $\xi^{-1}$  contribution (3.26).

**3.3. Final linearized bounds.**

**COROLLARY 3.4.** *Under assumptions (H1)–(H4), (D1)–(D3'), the Green function  $G(x, t; y)$  of (1.11) decomposes as  $G = E + \tilde{G}$ ,*

$$(3.28) \quad E = \bar{U}'(x)e(x, t; y),$$

where, for some  $C > 0$ , all  $t > 0$ ,  $1 \leq q \leq 2 \leq p \leq \infty$ ,  $0 \leq j, k, l$ ,  $j + l \leq K + 1$ ,  $1 \leq r \leq 2$ ,

$$(3.29) \quad \begin{aligned} & \left\| \int_{-\infty}^{+\infty} \tilde{G}(x, t; y) f(y) dy \right\|_{L^p(x)} \leq C(1+t)^{-\frac{1}{2}(1/q-1/p)} \|f\|_{L^q \cap H^1}, \\ & \left\| \int_{-\infty}^{+\infty} \partial_y^r \tilde{G}(x, t; y) f(y) dy \right\|_{L^p(x)} \leq C(1+t)^{-\frac{1}{2}(1/q-1/p) - \frac{1}{2}} \|f\|_{L^q \cap H^{r+1}}, \\ & \left\| \int_{-\infty}^{+\infty} \partial_t^r \tilde{G}(x, t; y) f(y) dy \right\|_{L^p(x)} \leq C(1+t)^{-\frac{1}{2}(1/q-1/p) - \frac{1}{2}} \|f\|_{L^q \cap H^{2r+1}}, \\ & \left\| \int_{-\infty}^{+\infty} \tilde{G}(x, t; y) (0, 1)^T f(y) dy \right\|_{L^p(x)} \leq C(1+t)^{-\frac{1}{2}(1/q-1/p) - \frac{1}{2}} \|f\|_{L^q \cap H^1}, \end{aligned}$$

$$(3.30) \quad \begin{aligned} & \left\| \int_{-\infty}^{+\infty} \partial_x^j \partial_t^k e(x, t; y) f(y) dy \right\|_{L^p} \leq (1+t)^{-\frac{1}{2}(1/q-1/p) - \frac{(j+k)}{2} + \frac{1}{2}} \|f\|_{L^q}, \\ & \left\| \int_{-\infty}^{+\infty} \partial_x^j \partial_t^k \partial_y^r e(x, t; y) f(y) dy \right\|_{L^p} \leq (1+t)^{-\frac{1}{2}(1/q-1/p) - \frac{(j+k)}{2}} \|f\|_{L^q}, \\ & \left\| \int_{-\infty}^{+\infty} \partial_x^j \partial_t^k e(x, t; y) (0, 1)^T f(y) dy \right\|_{L^p} \leq (1+t)^{-\frac{1}{2}(1/q-1/p) - \frac{(j+k)}{2}} \|f\|_{L^q}. \end{aligned}$$

Moreover,  $e(x, t; y) \equiv 0$  for  $t \leq 1$ .

*Proof.* (Case  $q = 1$ .) From (3.11) and the triangle inequality we obtain

$$\left\| \int_{\mathbb{R}} \tilde{G}^I(x, t; y) f(y) dy \right\|_{L^p(x)} \leq \int_{\mathbb{R}} \sup_y \|\tilde{G}^I(\cdot, t; y)\|_{L^p} |f(y)| dy \leq C(1+t)^{-\frac{1}{2}(1-1/p)} \|f\|_{L^1},$$

and similarly for  $y$ - and  $t$ -derivative estimates, and products with  $(0, 1)^T$ , which, together with (3.4), yield (3.29). Bounds (3.30) follow similarly by the triangle inequality and (3.12)–(3.13).

(Case  $q = 2$ .) From (3.17)–(3.18) and analyticity of  $v_j, \tilde{v}_j$ , we have boundedness from  $L^2([0, X]) \rightarrow L^2([0, X])$  of the projection-type operators

$$(3.31) \quad f \rightarrow \beta_{j,n} \tilde{\beta}_{j,l} \left( v_n(\xi, x) - \bar{U}'(x) \right) \langle \tilde{v}_l, f \rangle$$

and

$$(3.32) \quad f \rightarrow \beta_{j,k} \tilde{\beta}_{j,l} v_k(\xi, x) \langle \tilde{v}_l, f \rangle \quad \text{for } k \neq 1,$$



uniformly with respect to  $\xi$ , from which we obtain by (3.16), (3.25), and (1.16) the bound

$$(3.33) \quad \left\| \int_{-\infty}^{+\infty} \tilde{G}^I(x, t; y) f(y) dy \right\|_{L^2(x)} \leq C \|f\|_{L^2(x)}$$

for all  $t \geq 0$ , yielding, together with (3.4), the result (3.29) for  $p = 2, r = 1$ . Similarly, by boundedness of  $\tilde{v}_j, v_j, \bar{U}'$  in all  $L^p[0, X]$ , we have

$$\begin{aligned} \left\| e^{\lambda_j(\xi)t} \beta_{j,n} \tilde{\beta}_{j,l} \left( v_n(\xi, x) - \bar{U}'(x) \right) \langle \tilde{v}_l, \hat{f} \rangle \right\|_{L^\infty(x)} &\leq C e^{-\theta|\xi|^2 t} \|\hat{f}(\xi, \cdot)\|_{L^2(x)}, \\ \left\| e^{\lambda_j(\xi)t} \beta_{j,k} \tilde{\beta}_{j,l} v_k(\xi, x) \langle \tilde{v}_l, \hat{f} \rangle \right\|_{L^\infty(x)} &\leq C e^{-\theta|\xi|^2 t} \|\hat{f}(\xi, \cdot)\|_{L^2(x)} \quad \text{for } k \neq 1, \end{aligned}$$

$C, \theta > 0$ , yielding by definitions (3.16), (3.25) the bound

$$(3.34) \quad \begin{aligned} \left\| \int_{-\infty}^{+\infty} \tilde{G}^I(x, t; y) f(y) dy \right\|_{L^\infty(x)} &\leq \left( \frac{1}{2\pi} \right) \int_{-\pi}^{\pi} C \phi(\xi) e^{-\theta|\xi|^2 t} \|\hat{f}(\xi, \cdot)\|_{L^2(x)} d\xi \\ &\leq C \left\| \phi(\xi) e^{-\theta|\xi|^2 t} \right\|_{L^2(\xi)} \|\hat{f}\|_{L^2(\xi, x)} \\ &\leq C(1+t)^{-\frac{d}{4}} \|f\|_{L^2([0, X])}, \end{aligned}$$

hence giving the result for  $p = \infty, r = 0$ . The result for  $r = 0$  and general  $2 \leq p \leq \infty$  then follows by  $L^p$  interpolation between  $p = 2$  and  $p = \infty$ . Derivative bounds  $1 \leq r \leq 2$  follow by similar arguments, using (3.19)–(3.21), as do bounds for products with  $(0, 1)^T$ . Bounds (3.30) follow similarly.

(Case  $1 \leq q \leq 2$ .) By Riesz–Thorin interpolation between the cases  $q = 1$  and  $q = 2$ , we obtain the bounds asserted in the general case  $1 \leq q \leq 2, 2 \leq p \leq \infty$ .  $\square$

Note the close analogy between the bounds of Corollary 3.4 and those obtained in [15, 13] for the viscous or relaxation shock wave case.

**4. Nonlinear stability.** With the bounds of Corollary 3.4, nonlinear stability follows by a combination of the argument of [8, 9] and modifications introduced in the shock wave case to treat partial parabolicity and potential loss of derivatives in the nonlinear iteration scheme [32, 34].

**4.1. Nonlinear perturbation equations.** Given a solution  $\tilde{U}(x, t)$  of (1.2), define the nonlinear perturbation variable

$$(4.1) \quad v = U - \bar{U} = \tilde{U}(x + \psi(x, t), t) - \bar{U}(x),$$

where

$$(4.2) \quad U(x, t) := \tilde{U}(x + \psi(x, t), t)$$

and  $\psi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is to be chosen later.

LEMMA 4.1. For  $v, U$  as in (4.1), (4.2), and  $|\tilde{U}|$  bounded,

$$(4.3) \quad U_t + f(U)_x - (B(U)U_x)_x - g(U) = (\partial_t - L)\bar{U}'(x)\psi(x, t) + P + \partial_x R + \partial_t S,$$

where

$$(4.4) \quad P = \left( g(\tilde{U}) - g(\bar{U}) \right) \psi_x = (0, 1)^T \mathcal{O}(|v| |\psi_x|),$$

$$(4.5) \quad R := v\psi_t + B(\tilde{U})(\bar{U}_x + v_x) \frac{\psi_x^2}{1 + \psi_x} - \left( B(\tilde{U}) - B(\bar{U}) \right) \bar{U}_x \psi_x - B(\tilde{U})v_x \psi_x,$$

and

$$(4.6) \quad S := -v\psi_x.$$

*Proof.* By the definition of  $U$  in (4.2) we have by a straightforward computation

$$\begin{aligned} U_t(x, t) &= \tilde{U}_x(x + \psi(x, t), t)\psi_t(x, t) + \tilde{U}_t(x + \psi, t), \\ f(U(x, t))_x &= df(\tilde{U}(x + \psi(x, t), t))\tilde{U}_x(x + \psi, t) \cdot (1 + \psi_x(x, t)), \\ U_x(x, t) &= \tilde{U}_x(x + \psi(x, t), t) \cdot (1 + \psi_x(x, t)). \end{aligned}$$

By  $\tilde{U}_t + df(\tilde{U})\tilde{U}_x - (B(\tilde{U})\tilde{U}_x)_x - g(\tilde{U}) = 0$ , it follows that

$$(4.7) \quad \begin{aligned} U_t + f(U)_x - (B(U)U_x)_x - g(U) &= \tilde{U}_x\psi_t + df(\tilde{U})\tilde{U}_x\psi_x - (B(\tilde{U})\tilde{U}_x)_x\psi_x - (B(\tilde{U})\tilde{U}_x\psi_x)_x \\ &= \tilde{U}_x\psi_t - \tilde{U}_t\psi_x + g(\tilde{U})\psi_x - (B(\tilde{U})\tilde{U}_x\psi_x)_x, \end{aligned}$$

where it is understood that derivatives of  $\tilde{U}$  appearing on the right-hand side are evaluated at  $(x + \psi(x, t), t)$ . Moreover, by another direct calculation, using  $L(\bar{U}'(x)) = 0$ , we have

$$\begin{aligned} (\partial_t - L)\bar{U}'(x)\psi &= \bar{U}_x\psi_t - \bar{U}_t\psi_x + df(\bar{U})\bar{U}_x\psi_x - (B(\bar{U})\bar{U}_x)_x\psi_x - (B(\bar{U})\bar{U}_x\psi_x)_x \\ &= \bar{U}_x\psi_t - \bar{U}_t\psi_x + g(\bar{U})\psi_x - (B(\bar{U})\bar{U}_x\psi_x)_x. \end{aligned}$$

Subtracting, and using the facts that, by differentiation of  $(\bar{U} + v)(x, t) = \tilde{U}(x + \psi(x, t), t)$ ,

$$(4.8) \quad \bar{U}_x + v_x = \tilde{U}_x(1 + \psi_x), \quad \bar{U}_t + v_t = \tilde{U}_t + \tilde{U}_x\psi_t,$$

so that

$$(4.9) \quad \tilde{U}_x - \bar{U}_x - v_x = -(\bar{U}_x + v_x) \frac{\psi_x}{1 + \psi_x}, \quad \tilde{U}_t - \bar{U}_t - v_t = -(\bar{U}_x + v_x) \frac{\psi_t}{1 + \psi_x},$$

we obtain

$$(4.10) \quad \begin{aligned} U_t + f(U)_x - (B(U)U_x)_x - g(U) &= (\partial_t - L)\bar{U}'(x)\psi + v_x\psi_t - v_t\psi_x \\ &\quad + (g(\tilde{U}) - g(\bar{U}))\psi_x - \left( B(\tilde{U})v_x\psi_x \right)_x \\ &\quad + \left( B(\tilde{U})(\bar{U}_x + v_x) \frac{\psi_x^2}{1 + \psi_x} \right)_x \\ &\quad - \left( \left( B(\tilde{U}) - B(\bar{U}) \right) \bar{U}_x\psi_x \right)_x, \end{aligned}$$

yielding (4.3) by  $v_x\psi_t - v_t\psi_x = (v\psi_t)_x - (v\psi_x)_t$ .  $\square$

**COROLLARY 4.2.** *The nonlinear residual  $v$  defined in (4.1) satisfies*

$$(4.11) \quad v_t - Lv = (\partial_t - L)\bar{U}'(x)\psi - Q_x + T + P + R_x + \partial_t S,$$

where  $P$ ,  $R$ , and  $S$  are as in Lemma 4.1 and  $Q$  and  $T$  are defined by

$$(4.12) \quad \begin{aligned} Q &:= f(\tilde{U}(x + \psi(x, t), t)) - f(\bar{U}(x)) - df(\bar{U}(x))v \\ &\quad - \left( B(\tilde{U}(x + \psi(x, t), t))\tilde{U}_x(x + \psi(x, t), t) - B(\bar{U}(x))\bar{U}_x(x) \right) \\ &\quad - (B(\bar{U})v_x + (dB(\bar{U})\bar{U}_x)v) \end{aligned}$$

and

$$(4.13) \quad T := g(\bar{U}(x + \psi(x, t), t)) - g(\bar{U}(x)) - dg(\bar{U}(x))v = (0, 1)^T \mathcal{O}(|v|^2).$$

*Proof.* We obtain the proof by Taylor expansion comparing (4.3) and  $\bar{U}_t + f(\bar{U})_x - (B(\bar{U})\bar{U}_x)_x - g(\bar{U}) = 0$ .  $\square$

**4.2. Cancellation estimate.** Our strategy in writing (4.11) is motivated by the following basic cancellation principle.

PROPOSITION 4.3 (see [6]). *For any  $f(y, s) \in L^p \cap C^2$  with  $f(y, 0) \equiv 0$ , there holds*

$$(4.14) \quad \int_0^t \int G(x, t - s; y)(\partial_s - L_y)f(y, s)dy ds = f(x, t).$$

*Proof.* Integrating the left-hand side by parts, we obtain

$$(4.15) \quad \int G(x, 0; y)f(y, t)dy - \int G(x, t; y)f(y, 0)dy + \int_0^t \int (\partial_t - L_y)^*G(x, t - s; y)f(y, s)dy ds.$$

Noting that, by duality,

$$(\partial_t - L_y)^*G(x, t - s; y) = \delta(x - y)\delta(t - s),$$

$\delta(\cdot)$  here denoting the Dirac delta-distribution, we find that the third term on the right-hand side vanishes in (4.15), while, because  $G(x, 0; y) = \delta(x - y)$ , the first term is simply  $f(x, t)$ . The second term vanishes by  $f(y, 0) \equiv 0$ .  $\square$

**4.3. Nonlinear damping estimate.** The following technical result is a key ingredient in the nonlinear stability analysis that follows. Applying Duhamel’s principle to (4.11) and using Proposition 4.3 yields

$$(4.16) \quad v(x, t) = \int_{-\infty}^{\infty} G(x, t; y)v_0(y) dy + \int_0^t \int_{-\infty}^{\infty} G(x, t - s; y)(-Q_y + T + R_y + S_s)(y, s) dy ds + \psi(t)\bar{U}'(x).$$

Note that terms  $Q_y$  and  $S_s$  involve derivatives of  $v$  (respectively, second derivative in space and first derivative in time) of maximal order; hence to close a nonlinear iteration scheme based on (4.16) would appear to require delicate maximal regularity estimates rather than the straightforward ones that we have obtained. Indeed, estimated using the linearized bounds of Corollary 3.4, the right-hand side appears to lose several degrees of regularity as a function from  $H^K \rightarrow L^2$  of  $v$ . However, the next proposition, adapted from the methods of [16, 34], shows that higher-order derivatives are slaved to lower-order ones, and hence derivatives “lost” at the linearized level may be “regained” at the nonlinear level. This effectively separates the issues of decay and regularity, allowing us to close a nonlinear iteration without the use of maximal regularity estimates or a more complicated quasi-linear iteration scheme.

PROPOSITION 4.4. *Let  $v_0 \in H^K$  ( $K$  as in (H1)), and suppose that for  $0 \leq t \leq T$ , the  $H^K$  norm of  $v$  and the  $H^{K+1}$  norms of  $\psi_t(\cdot, t)$  and  $\psi_x(\cdot, t)$  remain bounded by a sufficiently small constant. Moreover, suppose that the Froude number  $F$ , viscosity  $\nu$ ,*

and velocity derivative  $\bar{u}_x$  satisfy the amplitude condition  $\nu\bar{u}_x < F^{-1}$ . Then there are constants  $C, \theta_1 > 0$  such that, for all  $0 \leq t \leq T$ ,

$$(4.17) \quad \|v(t)\|_{H^\kappa}^2 \leq C e^{-\theta_1 t} \|v(0)\|_{H^\kappa}^2 + C \int_0^t e^{-\theta_1(t-s)} (\|v\|_{L^2}^2 + \|(\psi_t, \psi_x)\|_{H^\kappa}^2)(s) ds.$$

The proof of this result will be given in Appendix A. Here, we briefly outline the main ideas. First, notice that by subtracting from (4.7) for  $U$  the equation for  $\bar{U}$ , we may write the nonlinear perturbation equation as

$$(4.18) \quad \begin{aligned} v_t + (Av)_x - (Bv_x)_x - Cv &= P - Q(v)_x + T(v) + \tilde{U}_x \psi_t - \tilde{U}_t \psi_x \\ &+ g(\bar{U})\psi_x - \left( B(\tilde{U})\tilde{U}_x \psi_x \right)_x, \end{aligned}$$

where  $A, B, C$  are as in (1.12),  $P, Q$ , and  $T$  are as in Corollary 4.2,  $g$  and  $B$  are as in (1.10), and it is understood that derivatives of  $\tilde{U}$  appearing on the right-hand side are evaluated at  $(x + \psi(x, t), t)$ . Using (4.9) to replace  $\tilde{U}_x$  and  $\tilde{U}_t$ , respectively, by  $\bar{U}_x + v_x - (\bar{U}_x + v_x) \frac{\psi_x}{1 + \psi_x}$  and  $\bar{U}_t + v_t - (\bar{U}_x + v_x) \frac{\psi_t}{1 + \psi_x}$ , and moving the resulting  $v_t \psi_x$  term to the left-hand side of (4.18), we obtain

$$(4.19) \quad \begin{aligned} (1 + \psi_x)v_t &= (Bv_x)_x - (Av)_x + Cv + P - Q(v)_x + T(v) \\ &+ (\bar{U}_x + v_x) \psi_t + g(\bar{U})\psi_x - \left( B(\tilde{U}) (\bar{U}_x + v_x) \frac{\psi_x}{1 + \psi_x} \right)_x. \end{aligned}$$

Define now the Friedrichs symmetrizer

$$(4.20) \quad \Sigma := \begin{pmatrix} 1 & 0 \\ 0 & \delta^{-2} \end{pmatrix},$$

where  $\delta^2 := -A_{12} = \bar{\tau}^{-3}(F^{-1} - 2\nu\bar{u}_x)$ . By (1.21),  $\Sigma$  is a symmetric positive definite symmetrizer for the hyperbolic part of (4.19) in the sense that  $\Sigma A = \begin{pmatrix} -c & -1 \\ -1 & -c\delta^{-2} \end{pmatrix}$  is a symmetric matrix, where  $A$  is as in (1.12). Furthermore, to compensate for the lack of total parabolicity of the governing equation, here indicated by the presence of a neutral eigenspace of the matrix  $\Sigma B$ , we introduce the skew-symmetric Kawashima compensator

$$(4.21) \quad K := \eta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad 0 < \eta \ll 1,$$

and note that, in particular, for  $\eta > 0$  sufficiently small there exists a constant  $\theta > 0$  such that  $\Re(KA + \Sigma B) \geq \theta$ .

Now defining the functional

$$\mathcal{E}[v] := \langle v, \Sigma v \rangle + \sum_{j=1}^K (\langle \partial_x^j v, K \partial_x^{j-1} v \rangle + \langle \partial_x^j v, \Sigma \partial_x^j v \rangle),$$

where here  $\langle \cdot, \cdot \rangle$  denotes the standard  $L^2(\mathbb{R}^n)$  inner product, we find by a direct but lengthy calculation using Sobolev embedding and interpolation to absorb nonlinear and intermediate-derivative terms that

$$(4.22) \quad \partial_t \mathcal{E}(v) \leq -\theta_1 \|v\|_{H^\kappa}^2 + C \left( \|v\|_{L^2}^2 + \|(\psi_t, \psi_x)\|_{H^\kappa(x,t)}^2 \right)$$

for some positive constants  $C, \theta_1 > 0$ , as long as  $\|\tilde{U}\|_{H^\kappa}$  remains bounded and the quantities  $\eta > 0$ ,  $\|v\|_{H^\kappa}$ , and  $\|(\psi_t, \psi_x)\|_{H^\kappa(x)}$  remain sufficiently small. By Cauchy–Schwarz and the fact that  $\Sigma$  is positive definite by (1.21), we have  $\mathcal{E}(v) \sim \|v\|_{H^\kappa}^2$  for  $\eta > 0$  sufficiently small, and hence (4.22) implies

$$\partial_t \mathcal{E}(v) \leq -\theta_1 \mathcal{E}(v) + C \left( \|v\|_{L^2}^2 + \|(\psi_t, \psi_x)\|_{H^\kappa(x,t)}^2 \right),$$

from which (4.17) follows by Gronwall’s inequality and, again, the equivalence of  $\mathcal{E}(v)$  and  $\|v\|_{H^\kappa}^2$ .

For more details and a complete proof of the key inequality (4.22), see Appendix A.

*Remark 9.* The condition (1.21) gives effectively an upper bound on the allowable amplitude of the wave, for fixed Froude number and viscosity. It is not clear that this has any connection with behavior. Certainly it is needed for our argument structure, and perhaps even for the validity of (4.17), which is itself convenient but clearly not necessary for stability. The resolution of this issue would be very interesting from the standpoint of applications, both in this and related contexts.

*Remark 10.* The Lagrangian formulation appears essential here in order to carry out the analysis. One can carry out damping estimates for sufficiently small-amplitude waves in Eulerian coordinates by the argument of [14] in the shock wave case; however, the large-amplitude argument of [16], depending on global noncharacteristicity of the wave—corresponding here to nonvanishing of  $u - s$ , where  $s$  is wave speed in Eulerian coordinates—together with bounded variation of  $\bar{U}_x$ , appears to fail irreparably in the periodic case. As we have shown here, the same argument succeeds in Lagrangian coordinates, provided that the linearized convection matrix  $A$  is symmetrizable (the meaning of bound (1.21)). For similar observations regarding the advantages for energy estimates of the special structure in Lagrangian coordinates, see [30].

**4.4. Integral representation/ $\psi$ -evolution scheme.** Recalling the Duhamel representation (4.16) of the perturbation  $v$  along with the decomposition  $G = \bar{U}'(x)e + \tilde{G}$  of Corollary 3.4, we find that defining  $\psi$  implicitly as

$$\begin{aligned} \psi(x, t) = & - \int_{-\infty}^{\infty} e(x, t; y) U_0(y) dy \\ & - \int_0^t \int_{-\infty}^{+\infty} e(x, t - s; y) (P - Q_y + T + R_y + S_s)(y, s) dy ds, \end{aligned} \tag{4.23}$$

where  $e$  is defined as in (3.25), results in the *integral representation*

$$\begin{aligned} v(x, t) = & \int_{-\infty}^{\infty} \tilde{G}(x, t; y) v_0(y) dy \\ & + \int_0^t \int_{-\infty}^{\infty} \tilde{G}(x, t - s; y) (P - Q_y + T + R_y + S_s)(y, s) dy ds \end{aligned} \tag{4.24}$$

for the nonlinear perturbation  $v$ ; see [32, 14] for further details. Furthermore, differentiating (4.23) with respect to  $t$ , and recalling that  $e(x, s; y) \equiv 0$  for  $s \leq 1$ ,

$$\begin{aligned} \partial_t^j \partial_x^k \psi(x, t) = & - \int_{-\infty}^{\infty} \partial_t^j \partial_x^k e(x, t; y) U_0(y) dy \\ & - \int_0^t \int_{-\infty}^{+\infty} \partial_t^j \partial_x^k e(x, t - s; y) (P - Q_y + T + R_y + S_s)(y, s) dy ds. \end{aligned} \tag{4.25}$$

Equations (4.24), (4.25) together form a complete system in the variables  $(v, \partial_t^j \psi, \partial_x^k \psi)$ ,  $0 \leq j, k \leq K + 1$ , from the solution of which we may afterward recover the shift  $\psi$  via (4.23). From the original differential equation (4.11), together with (4.25), we readily obtain short-time existence and continuity with respect to  $t$  of solutions  $(v, \psi_t, \psi_x) \in H^K$  by a standard contraction-mapping argument based on (4.17), (4.23), and (3.30).

**4.5. Nonlinear iteration.** Associated with the solution  $(U, \psi_t, \psi_x)$  of integral system (4.24)–(4.25), define

$$(4.26) \quad \zeta(t) := \sup_{0 \leq s \leq t} \|(v, \psi_t, \psi_x)\|_{H^K}(s)(1+s)^{1/4}.$$

LEMMA 4.5. *For all  $t \geq 0$  for which  $\zeta(t)$  is finite and sufficiently small, some  $C > 0$ , and  $E_0 := \|U_0\|_{L^1 \cap H^K}$  sufficiently small,*

$$(4.27) \quad \zeta(t) \leq C(E_0 + \zeta(t)^2).$$

*Proof.* By (4.4)–(4.6) and (4.12)–(4.13) and corresponding bounds on the derivatives together with definition (4.26),

$$(4.28) \quad \|(P, Q, R, S, T)\|_{L^1 \cap H^2} \leq \|(v, v_x, \psi_t, \psi_x)\|_{L^2}^2 + \|(v, v_x, \psi_t, \psi_x)\|_{H^2}^2 \leq C\zeta(t)^2(1+t)^{-\frac{1}{2}},$$

as long as  $|\psi_x| \leq |\psi_x|_{H^K} \leq \zeta(t)$  remains small. Applying Corollary 3.4 with  $q = 1$  to representations (4.24)–(4.25), we obtain for any  $2 \leq p < \infty$

$$(4.29) \quad \begin{aligned} \|v(\cdot, t)\|_{L^p(x)} &\leq C(1+t)^{-\frac{1}{2}(1-1/p)}E_0 \\ &\quad + C\zeta(t)^2 \int_0^t (1+t-s)^{-\frac{1}{2}(1/2-1/p)}(t-s)^{-\frac{3}{4}}(1+s)^{-\frac{1}{2}}ds \\ &\leq C(E_0 + \zeta(t)^2)(1+t)^{-\frac{1}{2}(1-1/p)} \end{aligned}$$

and

$$(4.30) \quad \begin{aligned} \|(\psi_t, \psi_x)(\cdot, t)\|_{W^{K+1,p}} &\leq C(1+t)^{-\frac{1}{2}}E_0 + C\zeta(t)^2 \int_0^t (1+t-s)^{-\frac{1}{2}(1-1/p)-1/2}(1+s)^{-\frac{1}{2}}ds \\ &\leq C(E_0 + \zeta(t)^2)(1+t)^{-\frac{1}{2}(1-1/p)}, \end{aligned}$$

yielding in particular that  $\|(\psi_t, \psi_x)\|_{H^{K+1}}$  is arbitrarily small, verifying the hypotheses of Proposition 4.4.<sup>11</sup> Using (4.17) and (4.29)–(4.30), we thus obtain  $\|v(\cdot, t)\|_{H^K(x)} \leq C(E_0 + \zeta(t)^2)(1+t)^{-\frac{1}{4}}$ . Combining this with (4.30),  $p = 2$ , rearranging, and recalling definition (4.26), we obtain (4.5).  $\square$

*Proof of Theorem 1.1.* By short-time  $H^K$  existence theory,  $\|(v, \psi_t, \psi_x)\|_{H^K}$  is continuous as long as it remains small, and hence  $\zeta$  remains continuous as long as it remains small. By (4.5), therefore, it follows by continuous induction that  $\zeta(t) \leq 2CE_0$  for  $t \geq 0$  if  $E_0 < 1/4C$ , yielding by (4.26) the result (1.22) for  $p = 2$ . Applying (4.29)–(4.30), we obtain (1.22) for  $2 \leq p \leq p_*$  for any  $p_* < \infty$ , with uniform constant  $C$ . Taking  $p_* > 4$  and estimating

$$\|P\|_{L^2}, \|Q\|_{L^2}, \|R\|_{L^2}, \|S\|_{L^2}, \|T\|_{L^2}(t) \leq \|(v, \psi_t, \psi_x)\|_{L^4}^2 \leq CE_0(1+t)^{-\frac{3}{4}}$$

<sup>11</sup>Note that we have gained a necessary one degree of regularity in  $\psi$ , the regularity of  $\psi$  being limited only by the regularity of the coefficients of the underlying PDE (1.2).

in place of the weaker (4.28), then applying Corollary 3.4 with  $q = 2$ , we finally obtain (1.22) for  $2 \leq p \leq \infty$  by a computation similar to (4.29)–(4.30); we omit the details of this final bootstrap argument. Estimate (1.23) then follows using (3.30) with  $q = 1$  by

$$(4.31) \quad \begin{aligned} \|\psi(t)\|_{L^p} &\leq CE_0(1+t)^{\frac{1}{2p}} + C\zeta(t)^2 \int_0^t (1+t-s)^{-\frac{1}{2}(1-1/p)}(1+s)^{-\frac{1}{2}} ds \\ &\leq C(1+t)^{\frac{1}{2p}}(E_0 + \zeta(t)^2), \end{aligned}$$

together with the fact that  $\tilde{U}(x, t) - \bar{U}(x) = v(x - \psi, t) + \bar{U}(x) - \bar{U}(x - \psi)$ , so that  $|\tilde{U}(\cdot, t) - \bar{U}|$  is controlled by the sum of  $|v|$  and  $|\bar{U}(x) - \bar{U}(x - \psi)| \sim |\psi|$ . This yields stability for  $|U - \bar{U}|_{L^1 \cap H^k}|_{t=0}$  sufficiently small, as described in the final line of the theorem.  $\square$

**Appendix A. Nonlinear energy estimate.** The goal of this appendix is to prove the inequality (4.22), which was the key ingredient in the nonlinear energy estimate in Proposition 4.4. To this end, we write the nonlinear perturbation equation (4.19) for the variable  $v = (\tau, u)^T$  as

$$(A.1) \quad (1 + \psi_x)v_t = (Bv_x)_x - (Av)_x + Cv + (\bar{U}_x + v_x)\psi_t + g(\bar{U})\psi_x + \mathcal{N},$$

where the function  $\mathcal{N} = \mathcal{N}(v, \bar{U}_x, \psi_x, \psi_t)$  is defined by

$$(A.2) \quad \mathcal{N} := P - Q(v)_x + T(v) - \left( B(\tilde{U}) (\bar{U}_x + v_x) \frac{\psi_x}{1 + \psi_x} \right)_x,$$

where  $P$ ,  $Q$ , and  $T$  are defined as in (4.4), (4.12), and (4.13), respectively. The key to the analysis is to carefully keep track of the “hyperbolic” ( $\tau$ ) and “parabolic” ( $u$ ) components of  $v$  separately. We begin by symmetrizing the linearized convection matrix  $A$  of (A.1) by introducing the Friedrichs symmetrizer  $\Sigma$  defined in (4.20) as

$$\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & \delta^{-2} \end{pmatrix}, \quad \delta^2 = \bar{\tau}^{-3} \left( \frac{1}{F} - 2\nu\bar{u}_x \right),$$

noting in particular that it is symmetric positive definite by the amplitude condition (1.21). The fact that  $\Sigma A = \begin{pmatrix} -c & -1 \\ -1 & -c\delta^{-2} \end{pmatrix}$  is symmetric then yields hyperbolic properties of the solution using straightforward energy estimates, integration by parts, and the Friedrichs symmetrizer relation

$$\Re \langle U, SU_x \rangle = -\frac{1}{2} \langle U, S_x U \rangle$$

valid for all self-adjoint operators  $S \in \mathbb{C}^{n \times n}$  and  $U \in \mathbb{C}^n$ . Furthermore, for convenience we provide here a list of the block-structure of the various matrices arising in the forthcoming proofs: notice by definition that

$$(A.3) \quad B, B_x = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}, \quad A_x, A_{xx} = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}, \quad C, C_x = \begin{pmatrix} 0 & 0 \\ & * \end{pmatrix},$$

which immediately yields

$$(A.4) \quad \Sigma B, \Sigma B_x, \Sigma_x B, \Sigma_{xx} B, \Sigma_x B_x = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix},$$

$$(A.5) \quad \Sigma A_x, \Sigma_x A, \Sigma_{xx} A, \Sigma_x A_x, \Sigma A_{xx} = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix},$$

and

$$(A.6) \quad \Sigma C = \Sigma C_x = \begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix}.$$

We will refer back to these observations throughout the proofs in this appendix.

*Remark 11.* The apparently special structure leading to (A.3) is in fact a special case of the more general structure pointed out in [30],<sup>12</sup> shared by the equations of one-dimensional gas dynamics, MHD, and viscoelasticity [30, 2] when expressed in Lagrangian coordinates.

Defining the first-order “Friedrichs bilinear form” as

$$\mathcal{F}_1[v_1, v_2] := \langle v_1, \Sigma v_2 \rangle + \langle \partial_x v_1, \Sigma \partial_x v_2 \rangle,$$

our first step in proving Proposition 4.4 is to establish the following lemma.

**LEMMA A.1.** *Let  $v(\cdot, 0) \in H^1$  and suppose that for  $0 \leq t \leq T$ , the  $H^1$  norm of  $v$  and the  $H^2$  norms of  $\psi_x$  and  $\psi_t$  remain bounded by a sufficiently small constant. Moreover, suppose that the amplitude condition (1.21) holds. Then we have the first-order “Friedrichs-type” estimate*

$$(A.7) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \mathcal{F}_1[v, v] &\leq -\langle v_x, w \Sigma B v_x \rangle - \langle v_{xx}, w \Sigma B v_{xx} \rangle \\ &+ C_1 \left( \|v\|_{L^2}^2 + \frac{1}{\varepsilon} \|u_x\|_{L^2}^2 + \varepsilon \|\tau_x\|_{L^2}^2 \right) \\ &+ C_2 \|\psi_x\|_{H^2} \left( \|v\|_{H^1}^2 + \|u_{xx}\|_{L^2}^2 \right) \\ &+ \frac{C_3}{\varepsilon} \left( \|\psi_t\|_{H^1}^2 + \|\psi_x\|_{H^1}^2 \right) + \mathcal{F}_1[v, w\mathcal{N}], \end{aligned}$$

valid for all  $0 \leq t \leq T$ , for some constants  $C_{1,2} > 0$  where  $w := (1 + \psi_x)^{-1} \in L^\infty$ .

*Proof.* First, notice that from (A.1) and the symmetry of  $\Sigma$  we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \langle v, \Sigma v \rangle &= \langle v, \Sigma v_t \rangle \\ &= \langle v, w \Sigma ((Bv_x)_x - (Av_x) + Cv + \bar{U}_x \psi_t + v_x \psi_t + g(\bar{U})\psi_x + \mathcal{N}) \rangle, \end{aligned}$$

where  $w := (1 + \psi_x)^{-1}$ . Since  $\Sigma B$  is symmetric by (A.4), then we have

$$\langle v, w \Sigma (Bv_x)_x \rangle = -\langle (w \Sigma_x) v, Bv_x \rangle - \langle v_x, w \Sigma B v_x \rangle = \frac{1}{2} \langle v, ((w \Sigma)_x B)_x v \rangle - \langle v_x, w \Sigma B v_x \rangle,$$

and similarly

$$\langle v, w \Sigma (Av)_x \rangle = \langle v, w \Sigma A_x v \rangle - \frac{1}{2} \langle v, (w \Sigma A)_x v \rangle.$$

Furthermore, assuming that  $\|\psi_t\|_{H^2}$  remains bounded, we clearly have the estimate

$$\begin{aligned} \langle v, w \Sigma ((\bar{U}_x + v_x)\psi_t + g(\bar{U})\psi_x) \rangle &= \langle v, w \Sigma \bar{U}_x \psi_t \rangle - \frac{1}{2} \langle v, (w \Sigma \psi_t)_x v \rangle + \langle v, w \Sigma g(\bar{U})\psi_x \rangle \\ &\lesssim (\|v\|_{L^2}^2 + \|\psi_t\|_{L^2}^2 + \|\psi_x\|_{L^2}^2), \end{aligned}$$

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<sup>12</sup>See conditions (A1)–(A2) of [30, 35].



which, by using Cauchy–Schwarz, immediately yields the zeroth-order estimate

$$\frac{1}{2} \frac{d}{dt} \langle v, \Sigma v \rangle \leq - \langle v_x, w \Sigma B v_x \rangle + C (\|v\|_{L^2}^2 + \|\psi_t\|_{L^2}^2 + \|\psi_x\|_{L^2}^2) + \langle v, w \Sigma \mathcal{N} \rangle$$

for some positive constant  $C > 0$ .

Continuing, we find that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \langle v_x, w \Sigma v_x \rangle &= \langle v_x, w_x \Sigma ((Bv_x)_x - (Av)_x + Cv + (\bar{U}_x + v_x) \psi_t + g(\bar{U}) \psi_x) \rangle \\ &\quad + \langle v_x, w \Sigma ((Bv_x)_{xx} - (Av)_{xx} + (Cv)_x + ((\bar{U}_x + v_x) \psi_t + g(\bar{U}) \psi_x)_x) \rangle \\ &\quad + \langle v_x, \Sigma (w \mathcal{N})_x \rangle \\ &=: I_1 + I_2 + \langle v_x, \Sigma (w \mathcal{N})_x \rangle. \end{aligned}$$

To estimate  $I_1$ , notice that (A.4) immediately yields

$$\langle v_x, w_x \Sigma (Bv_x)_x \rangle = \langle v_x, w_x \Sigma B_x v_x \rangle + \langle v_x, w_x \Sigma B v_{xx} \rangle \lesssim \|w_x\|_{L^\infty} \|u_x\|_{H^1}^2,$$

and that, similarly, we have the estimates

$$\langle v_x, w_x \Sigma (Av)_x \rangle, \quad \langle v_x, w_x \Sigma Cv \rangle \lesssim \|w_x\|_{L^\infty} \|v\|_{H^1}^2$$

by (A.5) and (A.6). Finally, noting that for  $\|\psi_t\|_{L^\infty}$  bounded we have

$$\langle v_x, w_x \Sigma ((\bar{U}_x + v_x) \psi_t + g(\bar{U}) \psi_x) \rangle \lesssim \|w_x\|_{L^\infty} (\|v\|_{H^1}^2 + \|\psi_t\|_{L^2}^2 + \|\psi_x\|_{L^2}^2),$$

we see that together these yield the estimate

$$I_1 \lesssim \|w_x\|_{L^\infty} (\|v\|_{H^1}^2 + \|u_{xx}\|_{L^2}^2 + \|\psi_t\|_{L^2}^2 + \|\psi_x\|_{L^2}^2).$$

To obtain the analogous estimate on  $I_2$ , first notice that (A.4) and the boundedness of  $\|w\|_{L^\infty}$ , together with Young’s inequality, imply

$$\begin{aligned} \langle v_x, w \Sigma (Bv_x)_{xx} \rangle &= - \langle (w \Sigma)_x v_x + w \Sigma v_{xx}, (Bv_x)_x \rangle \\ &= - \langle v_{xx}, w \Sigma B v_{xx} \rangle - \langle v_{xx}, w \Sigma B_x v_x \rangle \\ &\quad - \langle v_x, (w \Sigma)_x B_x v_x \rangle - \langle v_x, (w \Sigma)_x B v_{xx} \rangle \\ &\leq - \langle v_{xx}, w \Sigma B v_{xx} \rangle + \tilde{C}_1 \left( \frac{1}{\varepsilon} \|u_x\|_{L^2}^2 + \varepsilon \|u_{xx}\|_{L^2}^2 \right) \\ &\quad + \tilde{C}_2 \|w_x\|_{L^\infty} (\|u_x\|_{L^2}^2 + \|u_{xx}\|_{L^2}^2) \end{aligned}$$

for some constants  $\tilde{C}_1, \tilde{C}_2 > 0$ , where  $\varepsilon > 0$  is a sufficiently small constant to be chosen later. Similarly, using (A.5) and (A.6) we find that

$$\begin{aligned} \langle v_x, w \Sigma (Av)_{xx} \rangle &= \langle v_x, w \Sigma (A_{xx} v + A_x v_x) \rangle - \frac{1}{2} \langle v_x, (w \Sigma A)_x v_x \rangle \\ &\lesssim \|u\|_{H^1}^2 + \|w_x\|_{L^\infty} \|v\|_{H^1}^2, \\ \langle v_x, w \Sigma (Cv)_x \rangle &\lesssim \left( \|v\|_{L^2}^2 + \frac{1}{\varepsilon} \|u_x\|_{L^2}^2 + \varepsilon \|\tau_x\|_{L^2}^2 \right). \end{aligned}$$

Finally, noting again that  $\|\psi_t\|_{L^\infty}$  is bounded, we find that

$$\begin{aligned} \langle v_x, w \Sigma ((\bar{U}_x + v_x) \psi_t)_x \rangle &\lesssim \left( (\varepsilon + \|\psi_{xt}\|_{L^\infty}) \|v_x\|_{L^2}^2 + \frac{1}{\varepsilon} \|\psi_t\|_{H^1}^2 \right) - \frac{1}{2} \langle v_x, (w \Sigma \psi_t)_x v_x \rangle \\ &\lesssim (\varepsilon + \|w_x\|_{L^\infty} + \|\psi_{xt}\|_{L^\infty}) \|v_x\|_{L^2}^2 + \|u_x\|_{L^2}^2 + \frac{1}{\varepsilon} \|\psi_t\|_{H^1}^2 \end{aligned}$$

and

$$\langle v_x, w\Sigma(g(\bar{U})\psi_x)_x \rangle \lesssim \varepsilon \|v_x\|_{L^2}^2 + \frac{1}{\varepsilon} \|\psi_x\|_{H^1}^2.$$

Therefore, by choosing  $\|\psi_t\|_{H^2} < \varepsilon$ , so that  $\|\psi_{xt}\|_{L^\infty} < \varepsilon$  by Sobolev embedding, and noting that  $\|\psi_{xt}\|_{L^\infty}$  is bounded, we have

$$I_2 \leq -\langle v_{xx}, w\Sigma B v_{xx} \rangle + \tilde{C} \left( \|v\|_{L^2}^2 + \frac{1}{\varepsilon} \|u_x\|_{L^2}^2 + \varepsilon \|\tau_x\|_{L^2}^2 + \frac{1}{\varepsilon} \|\psi_t\|_{H^1}^2 + \frac{1}{\varepsilon} \|\psi_x\|_{H^1}^2 \right) + \|w_x\|_{L^\infty} (\|v\|_{H^1}^2 + \|u_{xx}\|_{L^2}^2)$$

for some constant  $\tilde{C} > 0$ , from which the lemma follows by noting that  $\|w_x\|_{L^\infty} \lesssim \|\psi_x\|_{H^2}$  by Sobolev embedding.  $\square$

From Lemma A.1 it follows that if the diffusion  $\Sigma B$  were positive definite, we would immediately have the bound

$$\frac{1}{2} \frac{d}{dt} \mathcal{F}_1[v, v] \leq -\theta \|v\|_{H^1}^2 + C_1 (\|v\|_{L^2}^2 + \|\psi_t\|_{H^1}^2 + \|\psi_x\|_{H^1}^2) + \mathcal{F}_1[v, w\mathcal{N}]$$

by using Sobolev embedding and choosing  $\varepsilon > 0$ ,  $\|\psi_x\|_{H^2}$  sufficiently small, which, up to the contribution of the nonlinear residual terms  $\mathcal{N}$ , has the form of the inequality stated in Proposition 4.4; see the proof of Lemma A.2 below for details on how this calculation would proceed. However, the lack of total parabolicity in the governing equation (1.2) is manifested here in the fact that the matrix  $\Sigma B$  is not positive definite, rather being only positive semidefinite with rank one. In order to compensate for this “degenerate diffusion,” we introduce the Kawashima compensator  $K$ , defined in (4.21) as

$$K := \eta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where  $0 < \eta \ll 1$  is a small parameter which will be determined later. The fact that the hyperbolic effects in (A.1) can compensate for this degeneracy in the diffusive term  $\Sigma B$  is the point of the following lemma.

**LEMMA A.2.** *Assume the amplitude condition (1.21) holds. Then for  $\eta > 0$  sufficiently small, the matrix  $\Sigma B + KA$  is positive definite and, furthermore, the associated bilinear form satisfies the coercivity estimate*

$$\langle \xi, (\Sigma B + KA) \xi \rangle \geq \theta (\|\xi_2\|_{L^2}^2 + \eta \|\xi_1\|_{L^2}^2)$$

for some constant  $\theta > 0$  and all  $\xi = (\xi_1, \xi_2)^T \in L^2(\mathbb{R})$ .

The proof of Lemma A.2 is based on a simple matrix perturbation argument and is omitted. Defining now the first-order “Kawashima bilinear form” as

$$\mathcal{E}_1[v_1, v_2] := \mathcal{F}_1[v_1, v_2] + \langle \partial_x v_1, K v_2 \rangle$$

and noting the special structures

$$(A.8) \quad KB, KB_x = \eta \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}, \quad KA_x = \eta \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}, \quad KC = \eta \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix},$$

we have the following refinement of the first-order Friedrichs-type estimate in Lemma A.1.

LEMMA A.3. *Under the same hypothesis as that of Lemma A.1 and for  $\eta > 0$  sufficiently small, we have the first-order “Kawashima-type” estimate*

$$\frac{1}{2} \frac{d}{dt} \mathcal{E}_1[v, v] \leq -\theta_1 (\|u_x\|_{H^1}^2 + \eta \|\tau_x\|_{L^2}^2) + \frac{C}{\eta^2} (\|v\|_{L^2}^2 + \|\psi_t\|_{H^1}^2 + \|\psi_x\|_{H^1}^2) + \mathcal{E}_1[v, w\mathcal{N}]$$

for some constants  $\theta_1, C > 0$ .

*Proof.* Using (A.8) along with arguments similar to those used in Lemma A.1, we obtain the estimate

$$\begin{aligned} \frac{1}{2} \partial_t \langle v_x, K v \rangle &= \langle v_x, w K ((Bv_x)_x - (Av)_x + Cv + \bar{U}_x \psi_t + v_x \psi_t + g(\bar{U}) \psi_x + \mathcal{N}) \rangle \\ &\leq C \eta \left( \frac{1}{\delta} \|v\|_{L^2}^2 + \delta \|\tau_x\|_{L^2}^2 + \frac{1}{\delta} \|u_x\|_{L^2}^2 + \frac{1}{\delta} \|u_{xx}\|_{L^2}^2 + \frac{1}{\delta} (\|\psi_t\|_{L^2}^2 + \|\psi_x\|_{L^2}^2) \right) \\ &\quad - \langle v_x, w K A v_x \rangle + \langle v_x, K w \mathcal{N} \rangle \end{aligned}$$

for some positive constants  $\theta_1, C > 0$ , and for any  $\delta > 0$  sufficiently small, where we have used that  $\|\psi_t\|_{L^\infty} \lesssim \|\psi_t\|_{H^1}$  can be chosen sufficiently small, say of order  $\mathcal{O}(\delta)$ . It follows then from Lemma A.1 that

$$\begin{aligned} \frac{1}{2} \partial_t \mathcal{E}_1[v, v] &\leq -\langle v_x, w (\Sigma B + KA) v_x \rangle - \langle v_{xx} w \Sigma B v_{xx} \rangle \\ &\quad + C_1 \left( \left( \frac{\eta}{\delta} + 1 \right) \|v\|_{L^2}^2 + \left( \frac{\eta}{\delta} + \frac{1}{\varepsilon} \right) \|u_x\|_{L^2}^2 + (\eta \delta + \varepsilon) \|\tau_x\|_{L^2}^2 + \frac{\eta}{\delta} \|u_{xx}\|_{L^2}^2 \right) \\ &\quad + C_2 \|\psi_x\|_{H^1} (\|v\|_{H^1}^2 + \|u_{xx}\|_{L^2}^2) \\ &\quad + C_3 \left( \frac{1}{\varepsilon} + \frac{1}{\delta} \right) (\|\psi_t\|_{H^1}^2 + \|\psi_x\|_{H^1}^2) + \mathcal{E}_1[v, w\mathcal{N}]. \end{aligned}$$

By Lemma A.2, then, we find that for  $\eta > 0$  sufficiently small, say  $0 < \eta < \eta_0$ , we have the estimate

$$-\langle v_x, w (\Sigma B + KA) v_x \rangle - \langle v_{xx} w \Sigma B v_{xx} \rangle \leq -\theta (\|u_x\|_{L^2}^2 + \eta \|\tau_x\|_{L^2}^2 + \|u_{xx}\|_{L^2}^2)$$

for some constant  $\theta > 0$ . Thus, by fixing  $\delta$  and choosing  $\varepsilon = \varepsilon(\eta)$  such that

$$(A.9) \quad 0 < \delta = \frac{\theta}{2C_1} \quad \text{and} \quad 0 < \varepsilon(\eta) = \frac{\eta\theta}{4},$$

we find that

$$(\eta(-\theta + C_1\delta) + \varepsilon) \|\tau_x\|_{L^2}^2 = -\frac{\theta\eta}{4} \|\tau_x\|_{L^2}^2.$$

By subsequently requiring that the free parameter  $\eta > 0$  satisfy

$$(A.10) \quad 0 < \eta \leq \min \left\{ \frac{\theta\delta}{2C_1}, \eta_0 \right\},$$

we similarly find that

$$\left( -\theta + \frac{\eta}{\delta} C_1 \right) \|u_{xx}\|_{L^2}^2 \leq -\frac{\theta}{2} \|u_{xx}\|_{L^2}^2,$$

from which it follows by the above requirements on the parameters  $\eta$ ,  $\varepsilon(\eta)$ , and  $\delta$  that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \mathcal{E}_1[v, v] &\leq -\hat{\theta} (\|u_x\|_{L^2}^2 + \eta \|\tau_x\|_{L^2}^2 + \|u_{xx}\|_{L^2}^2) \\ &\quad + \tilde{C}_1 \left( \|v\|_{L^2}^2 + \frac{1}{\eta} \|u_x\|_{L^2}^2 \right) + C_2 \|\psi_x\|_{H^1} (\|v\|_{H^1}^2 + \|u_{xx}\|_{L^2}^2) \\ &\quad + \frac{\tilde{C}_3}{\eta} (\|\psi_t\|_{H^1}^2 + \|\psi_x\|_{H^1}^2) + \mathcal{E}_1[v, w\mathcal{N}]. \end{aligned}$$

Next, the Sobolev inequality  $\|g_x\|_{L^2}^2 \leq \|g_{xx}\|_{L^2} \|g\|_{L^2}$ , along with Young’s inequality, implies that  $\frac{\tilde{C}_1}{\eta} \|u_x\|_{L^2}^2 \leq \frac{\theta_1}{2} \|u_{xx}\|_{L^2}^2 + \frac{2\tilde{C}_1}{\theta_1 \eta^2} \|u\|_{L^2}^2$ , which, by now choosing  $\|\psi_x\|_{H^1}^2$  sufficiently small so that  $-\frac{\hat{\theta}}{2} + C_2 \|\psi_x\|_{H^1}^2 < 0$ , completes the proof.  $\square$

Using similar arguments, we can obtain higher-order Kawashima-type estimates by defining the  $k$ th-order Kawashima bilinear form as

$$\mathcal{E}_k[v_1, v_2] := \langle v, \Sigma v \rangle + \sum_{j=1}^k (\langle \partial_x^j v_1, K \partial_x^{j-1} v_2 \rangle + \langle \partial_x^j v_1, \Sigma \partial_x^j v_2 \rangle)$$

for each  $k \in \mathbb{N}$ . Indeed, the following estimate can be obtained by simply iterating the above argument and using the Sobolev inequality  $\|g_x\|_{H^j} \leq \alpha \|\partial_x^{j+2} g\|_{L^2} + \alpha^{-1} \|g\|_{L^2}$  for  $\alpha > 0$  sufficiently small.

LEMMA A.4. *Let  $j \in \mathbb{N}$  and  $v(\cdot, 0) \in H^j$ , and suppose that for  $0 \leq t \leq T$ , the  $H^j$  norm of  $v$  and the  $H^{j+1}$  norms of  $\psi_x$  and  $\psi_t$  remain bounded by a sufficiently small constant. Moreover, suppose that condition (1.21) is satisfied. Then for  $\eta > 0$  sufficiently small there exist constants  $\theta_1, C > 0$  such that, for all  $0 \leq t \leq T$ ,*

$$\frac{1}{2} \frac{d}{dt} \mathcal{E}_j[v, v] \leq -\theta_1 (\|u_x\|_{H^j}^2 + \eta \|\tau_x\|_{H^{j-1}}^2) + \frac{C}{\eta^2} (\|v\|_{L^2}^2 + \|\psi_t\|_{H^j}^2 + \|\psi_x\|_{H^j}^2) + \mathcal{E}_j[v, w\mathcal{N}].$$

To complete the proof of Proposition 4.4 it remains to estimate the terms  $\mathcal{E}_j[v, w\mathcal{N}]$  corresponding to the nonlinear residual terms in the perturbation equation (A.1). In particular, our goal is to demonstrate that these terms can be absorbed into the bound already computed, in the sense that there exist constants  $C > 0$  and  $0 < \varepsilon \ll 1$  such that

$$(A.11) \quad \mathcal{E}_j[v, w\mathcal{N}] \leq \varepsilon (\|u_x\|_{H^j}^2 + \|\tau_x\|_{H^{j-1}}^2) + C (\|v\|_{L^2}^2 + \|\psi_t\|_{H^j}^2 + \|\psi_x\|_{H^j}^2).$$

To this end, we notice that from (4.4) we have the identity

$$P = \left( g(\tilde{U}) - g(\bar{U}) \right) \psi_x = \left( \int_0^1 dg(\bar{U} + \theta v) d\theta \right) v \psi_x.$$

Using Sobolev embedding, then, we can estimate  $P$  in  $H^1$  in a straightforward way, using that  $\|v\|_{L^\infty}$  is assumed to be small (say, at most one). Indeed, using the above integral representation for  $P$  we immediately obtain

$$\|P\|_{L^2} \lesssim \|v \psi_x\|_{L^2} \lesssim \|v\|_{L^2} \|\psi_x\|_{L^\infty} \lesssim \|v\|_{L^2} \|\psi_x\|_{H^1}$$

and similarly

$$\begin{aligned} \|P_x\|_{L^2} &\lesssim \left\| \left( \int_0^1 d^2g(\bar{U} + \theta v)d\theta \right) (\bar{U}_x + v_x)v\psi_x \right\|_{L^2} + \left\| \left( \int_0^1 dg(\bar{U} + \theta v)d\theta \right) (v\psi_x)_x \right\|_{L^2} \\ &\lesssim \|v\psi_x\|_{L^2} + \|v_x\psi_x\|_{L^2}\|v\|_{L^\infty} + \|(v\psi_x)_x\|_{L^2} \\ &\lesssim \|v\|_{L^2}\|\psi_x\|_{H^1} + \|v_x\|_{L^2}\|\psi_x\|_{H^1}\|v\|_{L^\infty} + \|v_x\|_{L^2}\|\psi_x\|_{H^1} + \|v\|_{L^\infty}\|\psi_{xx}\|_{L^2} \\ &\lesssim (\|v\|_{H^1} + \|v\|_{H^1}^2)\|\psi_x\|_{H^1} \\ &\lesssim \|v\|_{H^1}\|\psi_x\|_{H^1}. \end{aligned}$$

From these estimates, together with Cauchy–Schwarz, it follows that

$$\begin{aligned} \mathcal{E}_1[v, wP] &\lesssim \|v\|_{L^2}\|P\|_{L^2} + \|v_x\|_{L^2}\|P\|_{L^2} + \|v_x\|_{L^2}\|(wP)_x\|_{L^2} \\ &\lesssim \|v\|_{H^1}^2\|\psi_x\|_{H^1}, \end{aligned}$$

where, again, we have used the fact that  $\|\psi_x\|_{H^1}$  is small. Since we can control the size of  $\psi_x$  in  $H^1$ , it follows that the  $\|v_x\|_{L^2}$  term above can be absorbed in the sense that the above inequality is of the form (A.11).

Using similar arguments, we can express  $T(v)$  as

$$T(v) = g(\tilde{U}) - g(\bar{U}) - dg(\bar{U})v = \left( \int_0^1 \left( \int_0^1 d^2g(\bar{U} + \theta sv)ds \right) \theta v d\theta \right) v,$$

from which we get the estimates

$$\|T(v)\|_{L^2} \lesssim \|v\|_{L^\infty}\|v\|_{L^2} \lesssim \|v\|_{H^1}^2$$

and, similarly,

$$\|T(v)_x\|_{L^2} \lesssim \|v\|_{H^1}^2 + \|v\|_{H^1}^3 \lesssim \|v\|_{H^1},$$

where we have used the fact that  $\|v\|_{H^1}$  is small. As above, these estimates readily yield

$$\begin{aligned} \mathcal{E}_1[v, wT] &= \langle v, \Sigma wT \rangle + \langle v_x, KwT \rangle + \langle v_x, \Sigma(wT)_x \rangle \\ &\lesssim \|v\|_{L^2}\|v\|_{H^1}^2 + \|v_x\|_{L^2}\|v\|_{H^1}^2, \end{aligned}$$

which again absorbs due to the control over  $v$  in  $H^1$ .

To analyze the remaining terms of  $\mathcal{N}$ , consider the term  $(B(\tilde{U})(\bar{U}_x + v_x)\frac{\psi_x}{1+\psi_x})_x$  present at the end of (A.2). Using the representation  $B(\tilde{U}) = (\int_0^1 dB(\bar{U} + \theta v)d\theta)v + B(\bar{U})$  along with the smallness of  $v$  in  $H^1$  and  $\psi_x$  in  $H^1$ , it follows that the associated contribution to  $\mathcal{E}_1[v, w\mathcal{N}]$  can be absorbed as long as the highest-order Friedrichs term

$$\left\langle v_x, \Sigma \left( w \left( B(\tilde{U})(\bar{U}_x + v_x)\frac{\psi_x}{1+\psi_x} \right)_x \right) \right\rangle$$

can be shown to absorb. Using integration by parts, we have

$$\left\langle v_x, \Sigma \left( w \left( B(\tilde{U})(\bar{U}_x + v_x)\frac{\psi_x}{1+\psi_x} \right)_x \right) \right\rangle = - \left\langle (\Sigma v_x)_x, w \left( B(\tilde{U})(\bar{U}_x + v_x)\frac{\psi_x}{1+\psi_x} \right)_x \right\rangle,$$

which absorbs by estimates similar to those previously obtained, using the smallness of  $v$  in  $H^1$  and  $\psi_x$  in  $H^2$ .

To estimate the contribution of the final terms of  $\mathcal{E}_1[v, w\mathcal{N}]$ , associated with  $-Q_x$ , first write  $Q := Q_1 - Q_2 + Q_3$ , where

$$\begin{aligned} Q_1 &:= f(\tilde{U}) - f(\bar{U}) - df(\bar{U})v = \left( \int_0^1 \left( \int_0^1 d^2 f(\bar{U} + \theta sv) ds \right) \theta v \, d\theta \right) v, \\ Q_2 &:= B(\tilde{U})\tilde{U}_x - B(\bar{U})\bar{U}_x - B(\bar{U})v_x, \\ Q_3 &:= (dB(\bar{U})\bar{U}_x) v, \end{aligned}$$

and notice that the contribution of  $\mathcal{E}_1[v, w(Q_1 + Q_3)_x]$  absorbs using estimates analogous to those obtained above for  $P$  and  $T$ . To illustrate how to handle the contributions of  $Q_2$ , first notice that by (4.9) we have

$$Q_2 = \left( \int_0^1 dB(\bar{U} + \theta v) d\theta \, v \right) (\bar{U}_x + v_x) \frac{1}{1 + \psi_x} - B(\bar{U}) (\bar{U}_x + v_x) \frac{\psi_x}{1 + \psi_x},$$

which absorbs as above by the smallness of  $v$  in  $H^1$  and  $\psi_x$  in  $H^2$ .

From the above considerations, then, we immediately have the following lemma.

LEMMA A.5. *Under the same hypothesis as that of Lemma A.1, we have the first-order “Kawashima-type” estimate*

$$\frac{d}{dt} \mathcal{E}_1[v, v] \leq -\theta_1 (\|u_x\|_{H^1}^2 + \|\tau_x\|_{L^2}^2) + C (\|v\|_{L^2}^2 + \|\psi_x\|_{H^1}^2 + \|\psi_t\|_{H^1}^2),$$

valid for some constants  $\theta_1, C > 0$ .

By similar arguments, we may obtain an analogous  $H^m$  estimate (substituting everywhere  $H^m$  for  $H^1$  and  $H^{m-1}$  for  $L^2$ ) for any  $m \in \mathbb{N}$ , as in the statement of Lemma A.4. Finally, using one last time the Sobolev inequality  $\|g_x\|_{L^2}^2 \lesssim \|g_{xx}\|_{L^2} \|g\|_{L^2}$ , together with Young’s inequality, we have completed the proof of the key inequality (4.22), from which the proof of Proposition 4.4 follows.

**Appendix B. High-frequency resolvent bounds.** In this appendix, we carry out the high-frequency resolvent bounds needed for the high-frequency solution operator bounds of section 3.1. To begin, write

$$L_\xi = e^{-i\xi x} L e^{i\xi x} = \hat{\partial} B \hat{\partial} - \hat{\partial} A + C,$$

where  $\hat{\partial} := (\partial_x + i\xi)$ . Clearly, then, the norm  $\|f\|_{\hat{H}^1} := \|\hat{\partial} f\|_{L^2([0, X])} + \|f\|_{L^2([0, X])}$  is equivalent to the usual norm  $\|f\|_{H^1([0, X])}$  for  $\xi \in [-\pi, \pi]$  bounded. Further, note that, for periodic functions  $f, g$  on  $[0, X]$ , we have the usual integration by parts rule

$$(B.1) \quad \langle f, \hat{\partial} g \rangle = \langle -\hat{\partial} f, g \rangle,$$

where  $\langle \cdot, \cdot \rangle$  as above denotes the standard  $L^2$  complex inner product on  $[0, X]$ . The main result of this appendix is then that for  $|\xi|$  bounded away from zero and sufficiently small the resolvent operator  $(\lambda - L_\xi)^{-1}$  is uniformly  $H^1 \rightarrow H^1$  bounded for  $\Re(\lambda) = -\eta < -\theta < 0$  for some constant  $\theta > 0$ , which is the content of the following lemma.

LEMMA B.1. *Under the derivative condition (1.21), there exist constants  $C, R > 0$  and a constant  $\theta > 0$  sufficiently small such that for  $|\lambda| \geq R$  and  $\Re \lambda < -\theta$ ,*

$$(B.2) \quad \|w\|_{H^1([0, X])} \leq C \|(L_\xi - \lambda)w\|_{H^1([0, X])}.$$

*Proof.* For  $\Sigma, K$ , as defined in the proof of Proposition 4.4, define the first-order Kawashima–Bloch bilinear form as

$$\widehat{\mathcal{E}}_1[v_1, v_2] := \langle v_1, \Sigma v_1 \rangle + \langle \widehat{\partial} v_1, K v_2 \rangle + \langle \widehat{\partial} v_1, \Sigma \widehat{\partial} v_2 \rangle$$

and suppose  $w$  is a solution of  $(\lambda - L_\xi) w = f$ . Then using the coercivity estimate of Lemma A.2, it follows by taking the real part of the equation

$$\widehat{\mathcal{E}}_1[w, (\lambda - L_\xi) w] = \widehat{\mathcal{E}}_1[w, f]$$

and using the equivalence of  $\widehat{\mathcal{E}}_1[w, w] \sim \|w\|_{\widehat{H}^1}^2$  that, similarly as in the proof of Proposition 4.4, we obtain

$$(B.3) \quad (\Re \lambda + \tilde{\theta}) \|w\|_{\widehat{H}^1}^2 + \tilde{\theta} \|B \widehat{\partial}_x^2 w\|_{L^2}^2 \leq C(\|w\|_{L^2}^2 + \|f\|_{\widehat{H}^1}^2), \quad \tilde{\theta} > 0.$$

Similarly, defining the analogous first-order Friedrichs–Bloch bilinear form

$$\widehat{\mathcal{F}}_1[v_1, v_2] := \langle v_1, \Sigma v_1 \rangle + \langle \widehat{\partial} v_1, \Sigma \widehat{\partial} v_2 \rangle$$

and taking the imaginary part of the equation

$$\widehat{\mathcal{F}}_1[w, (\lambda - L_\xi) w] = \widehat{\mathcal{F}}_1[w, f],$$

we obtain

$$(B.4) \quad |\Im \lambda| \|w\|_{L^2}^2 \leq C(\|w\|_{\widehat{H}^1}^2 + \|B \widehat{\partial}_x^2 w\|_{L^2}^2 + \|f\|_{\widehat{H}^1}^2).$$

Summing (B.3) with a sufficiently small multiple of (B.4), we obtain for  $\Re \lambda > -\tilde{\theta}/2$

$$|\lambda| \|w\|_{\widehat{H}^1}^2 \leq C(\|w\|_{\widehat{H}^1}^2 + \|f\|_{\widehat{H}^1}^2),$$

yielding the result for  $|\lambda| > 2C$  by equivalence of  $H^1$  and  $\widehat{H}^1$ .  $\square$

**Appendix C. The subcharacteristic condition and Hopf bifurcation.** At equilibrium values  $u = \tau^{-1/2} > 0$ , the inviscid version

$$(C.1) \quad U_t + f(U)_x = \begin{pmatrix} 0 \\ q(U) \end{pmatrix}, \quad q(U) = 1 - \tau u^2, \quad f(U) = \begin{pmatrix} -u \\ \frac{1}{2F\tau^2} \end{pmatrix}$$

of (1.8) has hyperbolic characteristics equal to the eigenvalues  $\pm \frac{u^3}{\sqrt{F}}$  of  $df$ , and equilibrium characteristic  $\frac{u^3}{2}$  equal to  $\partial_\tau f(\tau, u_*(\tau))$ , where  $u_*(\tau) := \tau^{-1/2}$  is defined by  $q(\tau, u_*(\tau)) = 0$ . The subcharacteristic condition, i.e., the condition that the equilibrium characteristic speed lie between the hyperbolic characteristic speeds, is therefore

$$(C.2) \quad \frac{u^3}{2} < \frac{u^3}{\sqrt{F}},$$

or  $F < 4$  as stated in Remark 2.

For  $2 \times 2$  relaxation systems such as the above, the subcharacteristic condition is exactly the condition that constant solutions be linearly stable, as may be readily verified by computing the dispersion relation using the Fourier transform. For the full system (1.8) with viscosity  $\nu > 0$ , a similar computation, Taylor expanding the

dispersion relation about  $\xi = 0$ , reveals that constant solutions are stable with respect to *low-frequency perturbations* if and only if the subcharacteristic condition  $F < 4$  is satisfied.

Next, let us examine the profile ODE  $c^2\tau' + ((2F)^{-1}\tau^{-2})' = 1 - \tau(q - c\tau)^2 - c\nu(\tau^{-2}\tau')'$  near an equilibrium  $u_0 = (q - c\tau_0) = \tau_0^{-1/2} > 0$ , and examine the circumstances for which Hopf bifurcation occurs. Linearizing about  $\tau \equiv \tau_0$ , and rearranging, we obtain

$$(C.3) \quad c\nu\tau_0^{-2}\tau'' + (c^2 - c_s^2)\tau' + \left(\frac{u_0^3/2 - c}{u_0/2}\right)\tau, \quad c_s := \frac{u_0^3}{\sqrt{F}},$$

for which the eigenvalues are the roots  $\mu$  of  $\alpha\mu^2 + \beta\mu + \gamma = 0$ , where  $\alpha = c\nu\tau^{-2}$ ,  $\beta = c^2 - c_s^2$ , and  $\gamma = \frac{u_0^3/2 - c}{u_0/2}$ . Considering this as a problem indexed by parameters  $u_0$ ,  $c$ , and  $q$ , we see that Hopf bifurcation occurs when roots  $\mu_j(u_0, c, q)$  cross the imaginary axis as a conjugate pair, i.e., when  $\beta = 0$  and  $\gamma > 0$ .

These translate, using (C.2), to the *Hopf bifurcation conditions*

$$(C.4) \quad c = c_s = \frac{u_0^3}{\sqrt{F}} \quad \text{and} \quad F > 4.$$

Experiments of [17] indicate that bifurcation occurs at *minimum wave speed*, i.e., as  $c$  increases through the value  $c_s$ . That is, the minimum wave speed among nontrivial periodic waves is

$$(C.5) \quad c = c_s = \frac{u_0^3}{\sqrt{F}} = \frac{1}{\sqrt{F}\tau_0^3},$$

and the minimum value of  $F$  for which nontrivial periodic waves occur is  $F > 4$ . The frequency at bifurcation is

$$\omega = \sqrt{\gamma/\alpha} = \tau_0^{5/2}\nu^{-1/2}\sqrt{(\sqrt{F} - 2)},$$

and the period is  $X = \frac{2\pi}{\omega}$ .

So prescribing  $X$  as we do, we must choose  $F > 4$ , then solve  $\omega = \frac{2\pi}{X}$  to obtain

$$(C.6) \quad \tau_0 = \nu^{1/5} \left( \frac{4\pi^2}{X^2(\sqrt{F} - 2)} \right)^{1/5}.$$

Near this value and with  $c$  near  $c_s$ , we should find small-amplitude periodic waves.

*Remark 12.* The above discussion shows in passing that, similarly as observed in the conservative case in [20], small-amplitude periodic waves arising through Hopf bifurcation from constant solutions are necessarily *unstable* as solutions of the time-evolutionary PDE, since they inherit (a small perturbation of) the necessarily unstable dispersion relation of the limiting constant solution from which they bifurcate. On the other hand, in the large-amplitude limit, roll waves might well be stable. As observed by Gardner (see [5, 20]), this is determined by stability of the bounding homoclinic wave, which in the conservative case was known to be unstable. A good starting point for the study of roll waves, therefore, might be to determine linearized stability of *solitary pulse solutions* corresponding to homoclinic solutions of the profile ODE. Evidence for linearized stability of *some* viscous roll waves is given in [17], namely, the approximate Dressler waves arising in the small viscosity limit.



**Appendix D. Numerical stability investigation.** We conclude by suggesting a number of practical techniques for the numerical testing of stability. These can be carried out either in the Eulerian coordinates of [18] or in the Lagrangian coordinates of this paper. As suggested in a more general setting in [1], several of the algorithms may be easily adapted from an existing nonlinear evolution code. Comparison of these different methods and determination of stability in different regimes are interesting problems that we hope to carry out in future work.

**D.1. Method one: The power method.** The power method is an easy numerical method to approximate the function  $R(\xi) := \max \Re \sigma(L_\xi)$  determining stability, with  $R(\xi) < 0$  for  $\xi \neq 0$  corresponding to (D1) and  $R(\xi) \leq -\theta \xi^2$  corresponding to (D2). (Condition (D3) can be verified by an Evans analysis, as was already done in some cases in [18] and elsewhere.)

The method is just to approximate numerically the time evolution of the linearized equation  $w_t = L_\xi w$  on  $[0, X]$  with periodic boundary conditions, which should be a straightforward adaptation/simplification of nonlinear code presumably already written to study nonlinear stability with respect to periodic perturbations. Denote the solution operator as  $e^{L_\xi t}$ . Then a good approximation is

$$(D.1) \quad R(\xi) \approx T^1 \log \frac{|e^{L_\xi 2T} f|_{L^2}}{|e^{L_\xi T} f|_{L^2}},$$

where  $f$  is a square wave pulse centered at  $x = X/2$  and  $T$  is large, say  $T = 10, T = 50,$  or  $T = 100$ . This should be relatively straightforward, and plotting  $R(\xi)$  against  $\xi$  for  $\xi \in [-\pi, \pi]$  should quickly determine stability. See [3] for related investigations and discussion.

**D.2. Method two: Discretization.** Instead of Evans computations as in [20] (these involved finding the zero-level set of a two-parameter Evans function, with reported problematic results), one could alternatively proceed from a Bloch decomposition/matrix linear algebra point of view.

That is, one could discretize  $L_\xi$  on  $[0, X]$  with periodic boundary conditions as a large tridiagonal matrix

$$(D.2) \quad T(\xi) := (\Delta + i\xi)B(\Delta + i\xi) - (\Delta + i\xi)A + C,$$

acting on vectors  $(U_1, \dots, U_L)$  of sample points, where  $U_j \approx U(Xj/L)$ , and virtual point  $U_0 \equiv U_L$  (periodicity), and  $\Delta$  is a discrete derivative, for example, the forward difference over  $h := X/L$ , treating  $[0, X]$  as a torus to generate needed values  $U_j$  for  $j \leq 0$  or  $j > L$ . For each  $\xi$ , one may then call the fast linear algebra functions in MATLAB to generate the real part of the largest real part eigenvalue of  $T$  as a function  $R(\xi)$ . If  $R(\xi) < 0$  and  $R(\xi) \leq -c\xi^2, c > 0$ , then we have spectral stability—otherwise not. This should be fast even for a  $100 \times 100$  matrix or so. The discretization in  $\xi$  is over  $[-\pi, \pi]$ , so this is also no problem: 50 points should suffice.

Note [1] that discretization of the linearized operator  $L$  is typically already done for a standard method-of-lines realization of the linearized time evolution.

*Remark 13.* In an interesting recent talk by Barkley [1], he pointed out that using the power method for  $e^{Lt}, t$  small,<sup>13</sup> with an implicit scheme is something like using the inverse power method on  $(I - Lt)^{-1}$ . Note that  $(I - Lt)^{-1}$  is expected to be compact for  $t$  small in parabolic problems, so this is a preconditioning step paralleling the Fredholm theory or Birman–Schwinger approach on the analytical side.

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<sup>13</sup>This is essentially equivalent to the method suggested in Appendix D.1.

**D.3. Method three: Nonlinear evolution.** The simplest test of course is just to run the full nonlinear problem on a large domain  $[-NX, NX]$ ,  $N \gg 1$ , with periodic boundary conditions and square pulse wave initial conditions centered at  $x = 0$ . If the difference between the solution and the unperturbed periodic wave remains bounded in  $L^\infty$ , then the wave is stable, otherwise not. The experiment should be run only up to time  $T \ll NX$  to avoid interactions with the boundary. This and the sensitivity of numerical evolution of nonlinear equations are the main disadvantages of the method. The advantage is that this can be converted from existing nonlinear code for evolution on a single period  $[0, X]$  (easy to change). A variation is to solve the linearized equations  $v_t = Lv := (\partial_x B \partial_x - \partial_x A + C)v$  numerically, which would be more stable but require modification (straightforward, however) of the nonlinear code, changing over to linear.

**D.4. Method four: Evans function computations.** A final approach is to compute the Evans function  $D(\xi, \lambda)$  (straightforward [20, 18], but not particularly numerically well-conditioned) and plot zero-level sets of  $D(\xi, \cdot)$  for varying  $\xi$  (harder). This is not recommended in the basic form just described; in practice this was time-intensive and gave poorly resolved results [20]. A somewhat more reasonable variation would be to plot just the level sets near  $(\xi, \lambda) = (0, 0)$  (difficult, due to crossing/singularity at the origin, but contained) to verify (D2), then use winding number computations for  $D(\xi, \cdot)$  to verify (D1).

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