On the Modulation Equations and Stability of Periodic GKdV Waves via Bloch Decompositions*

Mathew A. Johnson[†], Kevin Zumbrun[‡] & Jared C. Bronski[§]

July 28, 2010

Keywords: Modulational Instability; Periodic Waves; Generalized Korteweg-de Vries Equation; Whitham Equations

Abstract

In this paper, we consider the relation between Evans function based approaches to the stability of periodic travelling waves and other theories based on long wavelength asymptotics together with Bloch wave expansions. In previous work it was shown by rigorous Evans function calculations that the formal slow modulation approximation resulting in the linearized Whitham averaged system accurately describes the spectral stability to long wavelength perturbations. To clarify the connection between Bloch-wave based expansions and Evans function based approaches, we reproduce this result without reference to the Evans function by using direct Bloch-expansion methods and spectral perturbation analysis. One of the novelties of this approach is that we are able to calculate the relevant Bloch waves explicitly for arbitrary finite amplitude solutions. Furthermore, this approach has the advantage of applying in the more general multi-periodic setting where no conveniently computable Evans function is yet devised.

1 Introduction

In this paper, we consider traveling wave solutions of the generalized Korteweg-de Vries (GKdV) equation

$$u_t = u_{xxx} + f(u)_x \tag{1.1}$$

*This is a preprint of an article accepted for publication in Physica D. Electronic publishing information will be made available as it becomes available.

[†]Indiana University, Bloomington, IN 47405; matjohn@indiana.edu: Research of M.J. was partially supported by an NSF Postdoctoral Fellowship under NSF grant no. DMS-0902192.

[‡]Indiana University, Bloomington, IN 47405; kzumbrun@indiana.edu: Research of K.Z. was partially supported under NSF grants no. DMS-0300487 and DMS-0801745.

[§]University of Illinois at Urbana-Champaign, Urbana, IL 61801; jared@math.uiuc.edu: Research of J.C.B. was partially supported under NSF grants no. NSF DMS 0807584.

where u is a scalar, $x, t \in \mathbb{R}$, and $f \in C^2(\mathbb{R})$ is a suitable nonlinearity. Equations of this form arise in a variety of applications and are equally interesting from a mathematical point of view. If f is a polynomial of degree three or less then the equation is exactly integrable via the inverse scattering transform. For example, in the case $f(u) = u^2$ equation (1.1) reduces to the classical KdV equation, a standard model of unidirectional long wave propagation in a channel[KdV, SH]. The cubic case $f(u) = \beta u^3 + \alpha u^2$, the modified KdV (mKdV) equation or Gardner equation, arises as a model for large amplitude internal waves in a density stratified medium, as well as a continuum approximation to bistable Fermi–Pasta-Ulam lattices[BM] [BO]. In each of these two cases, the corresponding PDE can be realized as a compatibility condition for a particular Lax pair and hence the corresponding Cauchy problem can (in principle) be completely solved via the famous inverse scattering transform. In the integrable case there is an extensive and very beautiful literature on the Whitham equations for KdV/mKdV (see Lax and Levermore [LL], Flashka, Forest and McLaughlin[FFM] and more recent references).

However, there are a variety of applications in which equations of form (1.1) arise which are not completely integrable and hence the inverse scattering transform can not be applied. For example, in applications in plasma physics equations of the form (1.1) arise with a wide variety of nonlinearities depending on particular physical considerations [KD] [M] [MS]. In this paper we study the stability for equations of KdV type for very general nonlinearities making use only of the integrability of the ordinary differential equation governing the traveling wave profiles. This ODE is, after an integration, Hamiltonian regardless of the integrability of the corresponding PDE.

It is well known that (1.1) admits traveling wave solutions of the form

$$u(x,t) = u_c(x + ct) \tag{1.2}$$

for $c > 0$. Historically, there has been much interest in the stability of traveling waves of the form (1.2) where the profile u_c decays exponentially to zero as its argument becomes unbounded, solutions known as solitary waves. The generalized KdV (1.1) admits solitary waves solutions and their stability is well understood and dates back to the pioneering work of Benjamin [B], which was then further developed by Bona, Grillakis, Shatah, Strauss, Pego, Weinstein and many others.

In general, however, the traveling waves u_c are periodic with the solitary waves forming a (typically co-dimension one) subset. In this paper we consider the stability of the periodic traveling wave solutions of (1.1). Compared with the solitary wave theory, the stability theory of periodic traveling waves is much less developed. Existing rigorous results in general fall into two categories: spectral stability with respect to localized or periodic perturbations [BrJ, BD], or nonlinear (orbital) stability with respect to periodic perturbations [BrJK, J1, DK]. Additionally there is a literature of using long-wavelength expansions, often formal and often based on a Bloch wave decomposition, to attempt to understand the stability.

In this paper we study the modulational stability by using a direct Bloch wave expansion of the linearized eigenvalue problem. This strategy is in some sense a combination of formal modulation expansions based on Bloch wave decompositions and rigorous analysis based on the Evans function and offers several advantages. One advantage is that Evans function techniques do not extend to multiple dimensions in a straight-forward way. In particular, these techniques seem not to apply to study the stability of multiply periodic solutions, that is, solutions which are periodic in more than one linearly independent spatial directions. Such equations are prevalent in the context of viscous conservation laws in multiple dimensions. In contrast, the Bloch expansion approach of the current work is based on spectral perturbation analysis and hence is not only completely rigorous but also generalizes to all dimensions with no difficulty. Secondly, the low-frequency Evans function analysis required to determine modulational instability is often quite tedious and difficult; see, for example, the analysis in the viscous conservation law case in [OZ1, OZ3, OZ4, Se1]. In contrast, however, we find that the Bloch-expansion methods of this paper are much more straightforward and reduces the problem to elementary matrix algebra. We should note, however, that although the our approach is simpler than the Evans calculation given in [BrJ], it is still nontrivial as a Bloch computation due to the presence of a Jordan block at the origin; notice this situation does not occur in the context of reaction diffusion equations where Bloch perturbation methods are widely used. Using this approach, then, we reproduce the modulational stability results of Bronski and Johnson without any specific mention or use of the Evans function. While a Bloch wave expansion is a very powerful and classical tool it often suffers from the fact that the Bloch waves are typically not known explicitly. This means the results are frequently evaluated in some asymptotic limit, commonly the small amplitude limit (see [DK,GH,Ha,HK,HaLS] for example). See also the classic work of Rowlands [Ro] in which similar ideas were used to analyze the long-wavelength stability of periodic solutions in the context of the nonlinear Schrödinger equation. In the current paper, following [BrJ], we are able to use Noether's theorem to explicitly generate the relevant Bloch waves and thus carry out the calculation for solutions of arbitrary amplitude.

Given the above remarks, we choose to carry out our analysis first in the gKdV case since, quite surprisingly, all calculations can be done very explicitly and can be readily compared to the results of [BrJ]. Our hope is that the present analysis may serve as a blueprint of how to consider the stability multiply periodic structures in more general, and difficult, equations than simply the gKdV equation. We should remark, however, that a slight variation of our approach was attempted in the latter half of the analysis in [BrJ]. However, the analysis was quite cumbersome requiring the asymptotic tracking of eigenfunctions which collapse to a Jordan block. Our approach does not require such delicate tracking and it is seen that we only need to asymptotically construct the projections onto the total eigenspace, which requires only tracking of an appropriate basis not necessarily consisting of eigenfunctions.

Finally, our approach connects in a very interesting way to the recent work of Johnson and Zumbrun [JZ1]. There, the authors studied the modulational stability of periodic gKdV waves in the context of Whitham theory; a well developed (formal) physical theory for dealing with such stability problems. It is important to note, however, that Whitham's approach to the KdV equation in [W] does not readily extend to the case of the gKdV considered here; indeed, in [W] it was assumed that the underlying periodic wave had zero mean in order to yield a closed system. While this assumption is always justified for the KdV equation by Galilean invariance, for more general nonlinearities in (1.1) the profile equation (2.1) admits periodic wave with nonzero mean and hence Whitham's method does not apply. Instead, the analysis in [JZ1] followed the work of Serre [Se1] by rescaling the governing PDE via the change of variables $(x, t) \mapsto (\varepsilon x, \varepsilon t)$ and then uses a WKB approximation of the solution to find a homogenized system which describes the mean behavior of the resulting approximation. In particular, it was found that a necessary condition for the stability of such solutions is the hyperbolicity i.e., local well-posedness of the resulting first order system of partial differential equations by demonstrating that the characteristic polynomial of the linearized Whitham averaged system accurately describes, to first order, the linearized dispersion relation arising in the Evans function analysis in [BrJ]. As seen in the recent work of [JZ4] and [JZ5], there is a deeper analogy between the low-frequency linearized dispersion relation and the Whitham averaged system at the structural level, suggesting a useful rescaling of the low-frequency perturbation problem. Moreover, it was seen that the Whitham modulation equations can provide invaluable information concerning not only the stability of nonlinear periodic solutions, but also their long-time behavior. It is this intuition that motivates our low-frequency analysis, and ultimately leads to our derivation of the Whitham averaged system for the gKdV using Bloch-wave expansions. In particular, this observation provides us with a rigorous method of justifying at a linearized level[1] the Whitham modulation equations which is independent of Evans function techniques used in [JZ1]. As mentioned above, such a result is desirable when trying to study the stability of multiply periodic waves where one does not have a useful notion of an Evans function.

The plan for the paper is as follows. In the next section we discuss the basic properties of the periodic traveling wave solutions of the gKdV equation (1.1), including a parametrization which we will find useful in our studies. We will then give a brief account of the results and methods of the papers [BrJ] and [JZ1] concerning the Evans function and Whitham theory approaches. We will then begin our analysis by discussing the Bloch decomposition of the linearized operator about the periodic traveling wave. In particular, we will show that the projection of the resulting Bloch eigenvalue problem onto the total eigenspace is a three-by-three matrix which is equivalent (up to similarity) to the linearized Whitham averaged system. We will then end with some concluding remarks and a discussion on consequences and open problems inferred from these results.

2 Preliminaries

Throughout this paper, we are concerned with spatially periodic traveling waves of the gKdV equation (1.1). To begin, we recall the basic properties of such solutions; for more information, see [BrJ] or [J1].

Traveling wave solutions of the gKdV equation with wave speed c are stationary solutions (1.1) in the moving coordinate frame $x + ct$ and whose profiles satisfy the traveling wave

[1] Inheriently, the Whitham equations are nonlinear and hence a full justification of the Whitham modulation equations for the gKdV would require analysis at the nonlinear level. This is an interesting open problem and is out of the scope of the current work.

ordinary differential equation

$$u_{xxx} + f(u)_x - cu_x = 0. \tag{2.1}$$

Clearly this equation is the derivative of a nonlinear oscillator equation and can be integrated up twice to bring the equation to quadrature

$$\frac{u_x^2}{2} = E + au + \frac{c}{2}u^2 - F(u) \tag{2.2}$$

where $F' = f$ and $F(0) = 0$, and a and E are constants of integration. Moreover, we assume that $F(u) = o(u^2)$ for $|u| \ll 1$. The existence of periodic orbits of (2.1) follows from the standard analysis; a necessary and sufficient condition is that the effective potential energy $V(u; a, c) := F(u) - \frac{c}{2}u^2 - au$ have a local minimum. It follows that the traveling wave solutions of (1.1) form a three parameter family of solutions parameterized by the constants a, E, and c (four parameter if one counts the translation mode). In particular on an open, not necessarily connected subset $\mathcal{D}$ of $(a, E, c) \in \mathbb{R}^3$ (2.2) has periodic solutions. On each connected component of $\mathcal{D}$ there are a fixed number of periodic solutions. The boundary of $\mathcal{D}$ corresponds to parameter values for which there are homoclinic/heteroclinic orbits and equilibrium solutions in addition to periodic solutions.

On the set $\mathcal{D}$ the turning point equation $E = V(u_\pm; a, c)$ has simple roots $u_\pm$ with $u_- < u_+$ and $E > V(u; a, c)$ for all $u \in (u_-, u_+)$. The roots $u_\pm$ are C^1 functions of the traveling wave parameters a, E, and c and, furthermore, to mod out translation invariance we can require $u(0; a, E, c) = u_-(a, E, c)$ and $u_x(0; a, E, c) = 0$ for all $(a, E, c) \in \mathcal{D}$, from which it follows that the function $u(x; a, E, c)$ is an even periodic solution of (2.1). Under these assumptions the functions $u_\pm$ are square root branch points of the function $\sqrt{E - V(u; a, c)}$. It follows that the period of the corresponding traveling wave solution of (2.1) is given by

$$T = T(a, E, c) = \frac{\sqrt{2}}{2} \oint_\Gamma \frac{du}{\sqrt{E - V(u; a, c)}}, \tag{2.3}$$

where integration over Γ represents a complete integration from u_- to u_+, and then back to u_-, with the negative branch of the square root chosen on the return trip. Alternatively, one can interpret Γ as a Jordan curve in the complex plane which encloses a bounded set containing both u_- and u_+. By a standard procedure, the above integral can be regularized at the square root branch points and hence represents a C^1 function of the traveling wave parameters on $\mathcal{D}$. Similarly, the conserved quantities of the gKdV flow can be expressed as

$$M = M(a, E, c) = \frac{\sqrt{2}}{2} \oint_\Gamma \frac{u\, du}{\sqrt{E - V(u; a, c)}} \tag{2.4}$$

$$P = P(a, E, c) = \frac{\sqrt{2}}{2} \oint_\Gamma \frac{u^2\, du}{\sqrt{E - V(u; a, c)}} \tag{2.5}$$

$$H = H(a, E, c) = \frac{\sqrt{2}}{2} \oint_\Gamma \frac{E - V(u; a, c) - F(u)}{\sqrt{E - V(u; a, c)}}\, du \tag{2.6}$$

where again these quantities, representing mass, momentum, and Hamiltonian, respectively, are finite and C^1 functions on $\mathcal{D}$. As seen in [BrJ, BrJK, J1], the gradients of the functions $T, M, P : \mathcal{D} \to \mathbb{R}$ play an important role in the modulational stability analysis of periodic traveling waves of the gKdV equation (1.1).

Remark 2.1. *Notice that in the derivation of the gKdV [BaMo], the solution u can represent either the horizontal velocity of a wave profile, or the density of the wave. Thus, the function $M : \mathcal{D} \to \mathbb{R}$ can properly be interpreted as a "mass" since it is the integral of the density over space. Similarly, the function $P : \mathcal{D} \to \mathbb{R}$ can be interpreted as a "momentum" since it is the integral of the density times velocity over space.*

To assist with calculations involving gradients of the above conserved quantities considered as functions on $\mathcal{D} \subset \mathbb{R}^3$, we note the following useful identity. The classical action (in the sense of action-angle variables) for the profile equation (2.1) is given by

$$K = \oint_\Gamma u_x \, du = \sqrt{2} \oint_\Gamma \sqrt{E - V(u; a, c)} du$$

where the contour Γ is defined as above. This provides a useful generating function for the conserved quantities of the gKdV flow restricted to the manifold of periodic traveling waves. Specifically, the classical action satisfies

$$T = \frac{\partial K}{\partial E}, \quad M = \frac{\partial K}{\partial a}, \quad P = 2\frac{\partial K}{\partial c} \tag{2.7}$$

as well as the identity

$$K = H + aM + \frac{c}{2}P + ET.$$

Together, these relationships imply the important relation

$$E\nabla_{a,E,c}T + a\nabla_{a,E,c}M + \frac{c}{2}\nabla_{a,E,c}P + \nabla_{a,E,c}H = 0.$$

So long as $E \neq 0$ then, gradients of the period, which is not itself conserved, can be interchanged with gradients of the genuine conserved quantities of the gKdV flow. As a result, all gradients and geometric conditions involved in the results in this paper can be expressed *completely* in terms of the gradients of the conserved quantities of the gKdV flow, which seems to be desired from a physical point of view. However, as the quantities T, M, and P arise most naturally in the analysis, we shall state our results in terms of these quantities alone.

We now discuss our main assumptions concerning the parametrization of the family of periodic traveling wave solutions of (1.1). To begin, we assume throughout this paper the period is not at a critical point in the energy, i.e. that $T_E \neq 0$. In other words, we assume that the period serves as a good local coordinate for nearby waves on $\mathcal{D}$ with fixed wave speed $c > 0$ and parameter a. As seen in [BrJK, J1], such an assumption is natural from the viewpoint of nonlinear stability. Moreover, we assume the period and mass provide good local coordinates for the periodic traveling waves of fixed wave speed $c > 0$. More precisely, given $(a_0, E_0, c_0) \in \mathcal{D}$ with $c_0 > 0$, we assume the map

$$(a, E) \mapsto (T(a, E, c_0), M(a, E, c_0))$$

have a unique C^1 inverse in a neighborhood of $(a_0, E_0) \in \mathbb{R}^2$, which is clearly equivalent with the non-vanishing of the Jacobian

$$\det\left(\frac{\partial(T, M)}{\partial(a, E)}\right)$$

at the point (a_0, E_0, c_0). As such Jacobians will be prevalent throughout our analysis, for notational simplicity we introduce the following Poisson bracket notation for two-by-two Jacobians

$$\{f, g\}_{x,y} := \det\left(\frac{\partial(f, g)}{\partial(x, y)}\right)$$

and the corresponding notation $\{f, g, h\}_{x,y,z}$ for three-by-three Jacobians. Finally, we assume that the period, mass, and momentum provide good local coordinates for nearby periodic traveling wave solutions of the gKdV. That is, given $(a_0, E_0, c_0) \in \mathcal{D}$ with $c_0 > 0$, we assume that the Jacobian $\{T, M, P\}_{a,E,c}$ is nonzero. While these re-parametrization conditions may seem obscure, the non-vanishing of these Jacobians has been seen to be essential in both the spectral and non-linear stability analysis of periodic gKdV waves in [BrJ, J1, BrJK]. In particular, these Jacobians have been computed in [BrJK] for several power-law nonlinearities and, in the cases considered, has been shown to be generically nonzero. Moreover, such a nondegeneracy condition should not be surprising as a similar condition is often enforced in the stability theory of solitary waves (see [Bo, B, PW]).

Now, fix a $(a_0, E_0, c_0) \in \mathcal{D}$. Then the stability of the corresponding periodic traveling wave solution may be studied directly by linearizing the PDE

$$u_t = u_{xxx} + f(u)_x - cu_x \tag{2.8}$$

about the stationary solution $u(\cdot, a_0, E_0, c_0)$ and studying the $L^2(\mathbb{R})$ spectrum of the associated linearized operator

$$\partial_x \mathcal{L}[u] := \partial_x \left(-\partial_x^2 - f'(u) + c\right).$$

As the coefficients of $\mathcal{L}[u]$ are T-periodic, Floquet theory implies the L^2 spectrum is purely continuous and corresponds to the union of the L^∞ eigenvalues corresponding to considering the linearized operator with periodic boundary conditions $v(x + T) = e^{i\kappa}v(x)$ for all $x \in \mathbb{R}$, where $\kappa \in (-\pi, \pi]$ is referred to as the Floquet exponent and is uniquely defined mod 2π. In particular, it follows that $\mu \in \mathrm{spec}(\partial_x\mathcal{L}[u])$ if and only if $\partial_x\mathcal{L}[u]$ has a bounded eigenfunction v satisfying $v(x+T) = e^{i\kappa}v(x)$ for some $\kappa \in \mathbb{R}$. More precisely, writing the spectral problem $\partial_x\mathcal{L}[u]v = \mu v$ as a first order system

$$Y'(x) = H(x, \mu)Y(x), \tag{2.9}$$

and defining the monodromy map $\mathbb{M}(\mu) := \Phi(T)\Phi(0)^{-1}$, where Φ is a matrix solution of (2.9), it follows that $\mu \in \mathbb{C}$ belongs to the L^2 spectrum of $\partial_x \mathcal{L}[u]$ if and only if the periodic Evans function

$$D(\mu,\kappa) := \det\left(\mathbb{M}(\mu) - e^{i\kappa} I\right)$$

vanishes for some $\kappa \in \mathbb{R}$.

When studying the modulational stability of the stationary periodic solution $u(\cdot; a_0, E_0, c_0)$ it suffices to study the zero set of the Evans function at low frequencies, i.e. seek solutions of $D(\mu,\kappa) = 0$ for $|(\mu,\kappa)| \ll 1$. Indeed, notice that the low-frequency expansion $\mu(\kappa)$ for (μ,κ) near $(0,0)$ may be expected to determine the long-time behavior and can be derived by the lowest order terms of the Evans function in a neighborhood of $(0,0)$. As a first step in expanding D, notice by translation invariance of (2.8) it follows that

$$\partial_x \mathcal{L}[u] u_x = 0,$$

that is, u_x is in the right T-periodic kernel of the $\partial_x \mathcal{L}[u]$. It immediately follows that $D(0,0) = 0$, and hence to determine modulational stability we must find all solutions of the form $(\mu(\kappa),\kappa)$ of the equation $D(\mu,\kappa) = 0$ in a neighborhood of $\kappa = 0$. Using Noether's theorem, or appropriately differentiating the integrated profile equation (2.2), it follows that the functions u_a and u_E formally satisfy

$$\partial_x \mathcal{L}[u] u_a = \partial_x \mathcal{L}[u] u_E = 0, \quad \partial_x \mathcal{L}[u] u_c = -u_x.$$

However, these functions are not in general T-periodic due to the secular dependence of the period on the parameters a, E, and c. Nevertheless, one can take linear combinations of these functions to form another T-periodic null-direction and a T-periodic function in a Jordan chain above the translation direction. The key point here is this: since we are able to generate a three parameter family of periodic solutions the derivatives of these solutions generate all of the Bloch functions in the kernel of the linearized operator. A tedious, but fairly straightforward, calculation (see [BrJ]) now yields

$$D(\mu,\kappa) = \Delta(\mu,\kappa) + \mathcal{O}(|\mu|^4 + \kappa^4), \tag{2.10}$$

where Δ represents a homogeneous degree three polynomial of the variables μ and κ. Defining the projective coordinate $y = \frac{i\kappa}{\mu}$ in a neighborhood of $\mu = 0$, we find that the modulational stability of the underlying periodic wave $u(\cdot; a_0, E_0, c_0)$ may then be determined by the discriminant of the polynomial $R(y) := \mu^{-3}\Delta(1, -iy)$, which takes the explicit form

$$R(y) = -y^3 + \frac{y}{2}\left(\{T,P\}_{E,c} + 2\{M,P\}_{a,E}\right) - \frac{1}{2}\{T,M,P\}_{a,E,c}. \tag{2.11}$$

Modulational stability then corresponds with R having three real roots, while the presence of root with nonzero imaginary part implies modulational instability.

On the other hand, we may also study the stability of the solution $u(\cdot; a_0, E_0, c_0)$ through the formal approach of Whitham theory. Indeed, recalling the recent work of [JZ1] we find upon rescaling (1.1) by $(x,t) \mapsto (\varepsilon x, \varepsilon t)$ and carrying out a formal WKB expansion as $\varepsilon \to 0$ a closed form system of three averaged, or homogenized, equations of the form

$$\partial_t \langle M\omega, P\omega, \omega\rangle (\dot{u}) + \partial_x \langle F, G, H\rangle (\dot{u}) = 0 \tag{2.12}$$

which we refer to as the Whitham averaged system[2], where $\omega = T^{-1}$ and $\dot{u}$ represents the equivalence class (modulo spatial translation) of a periodic traveling wave solution in the vicinity of the underlying (fixed) periodic wave $u(\cdot; a_0, E_0, c_0)$. The problem of stability of $u(\cdot; a_0, E_0, c_0)$ to long-wavelength perturbations may heuristically expected to be related to the linearization of (2.12) about the constant solution $\dot{u} \equiv u(\cdot; a_0, E_0, c_0)$, provided the WKB approximation is justifiable by stability considerations. This leads one to the consideration of the homogeneous degree three linearized dispersion relation

$$\hat{\Delta}(\mu,\kappa) := \det\left(\mu\frac{\partial(M\omega, P\omega, \omega)}{\partial(\dot{u})} - \frac{i\kappa}{T}\frac{\partial\,(F,G,H)}{\partial(\dot{u})}\right)(u(\cdot; a_0, E_0, c_0)) = 0 \tag{2.13}$$

where μ corresponds to the Laplace frequency and κ the Floquet exponent.

The main result of [JZ1] was to demonstrate a direct relationship between the above approaches. In particular, the following theorem was proved.

Theorem 1 ([JZ1])**.** *Under the assumptions that the Whitham system* (2.12) *is of evolutionary type, i.e.* $\frac{\partial(M\omega,P\omega,\omega)}{\partial(\dot{u})}$ *is invertible at* $(a_0, E_0, c_0) \in \mathcal{D}$*, and that the matrix* $\frac{\partial(\dot{u})}{\partial(a,E,c)}$ *is invertible at* $(a_0, E_0, c_0) \in \mathcal{D}$*, we have that in a neighborhood of* $(\mu,\kappa) = (0,0)$ *the asymptotic expansion*

$$D(\mu,\kappa) = \Gamma_0\hat{\Delta}(\mu,\kappa) + \mathcal{O}(|\mu|^4 + \kappa^4)$$

for some constant $\Gamma_0 \neq 0$*.*

Remark 2.2. *The assumption in Theorem 1 that* $\frac{\partial(M\omega,P\omega,\omega)}{\partial(\dot{u})}$ *is invertible forces the Whitham averaged system* (2.12) *to be of evolutionary type. Moreover, notice that the Whitham averaged system is inherently a relation of functions of the variable* $\dot{u}$*, while the Evans function calculations from [BrJ] utilize a parametrization of the family of traveling wave solutions of* (2.1) *(modulo spatial translation) by the parameters* (a,E,c)*. In order to compare the two linearized dispersion relations* $\Delta(\mu,\kappa)$ *and* $\hat{\Delta}(\mu,\kappa)$ *then we must ensure that we may freely interchange between these variables, i.e. we must assume that the matrix* $\frac{\partial(\dot{u})}{\partial(a,E,c)}$ *is invertible at the underlying periodic wave.*

That is, up to a constant, the dispersion relation (2.13) for the homogenized system (2.12) accurately describes the low-frequency limit of the exact linearized dispersion relation described in (2.10). As a result, Theorem 1 may be regarded as a justification at a linearized level of the WKB expansion and the formal Whitham procedure as applied to the gKdV

[2] It is important to note these equations are not the Whitham equations for the gKdV, which are traditionally the full nonlinear equations resulting from a formal WKB expansion. As seen in [JZ1], a formal WKB expansion yields terms of divergence form which vanish upon averaging and results in the homogenized system (2.12). This motivates our referring to (2.12) as the Whitham *averaged* system.

equation. The importance of this result stems from our discussion in the previous section: although the Evans function techniques of [BrJ] do not extend in a straightforward way to multiperiodic waves, the formal Whitham procedure may still be carried out nonetheless. However, it follows that we must find a more robust method of study to justify the Whitham expansion in higher dimensional cases. The purpose of this paper is to present precisely such a method using Bloch-expansions of the linearized operator near zero-frequency. Indeed, we will show how this general method in the case of the gKdV equation may be used to easily rigorously reproduce the linearized dispersion relation $\hat{\Delta}(\mu,\kappa)$ corresponding to the Whitham averaged system as well as justifying at a linearized level the Whitham expansion beyond stability to the level of long-time behavior of the perturbation.

3 Bloch Decompositions and Modulational Stability

In this section, we detail the methods of our modulational stability analysis by utilizing Bloch-wave decompositions of the linearized problem. In particular, our goal is to provide a rigorous justification at the linearized level of the Whitham expansion described in the previous section without reference to the Evans function. To this end, recall from Floquet theory that any bounded eigenfunction v of $\partial_x\mathcal{L}[u]$ must satisfy

$$v(x+T)=e^{i\kappa}v(x)$$

for some $\kappa\in(-\pi,\pi]$. Defining $\varepsilon=\varepsilon(\kappa)=\frac{i\kappa}{T}$, it follows that v must be of the form

$$v(x)=e^{\varepsilon x}P(x)$$

for some $\kappa\in(-\pi,\pi]$ and some T-periodic function P which is an eigenfunction of the corresponding operator

$$L^{\varepsilon}=e^{-\varepsilon x}\partial_x\mathcal{L}[u]e^{\varepsilon x}. \tag{3.1}$$

Notice that L^{ε} can be expanded as

$$L^{\varepsilon}=L_0+\varepsilon L_1+\varepsilon^2L_2-\varepsilon^3$$

where $L_0:=\partial_x\mathcal{L}[u]$ is the original linearized operator, $L_1:=\mathcal{L}[u]-2\partial_x^2$, and $L_2:=-3\partial_x$. With this motivation, we introduce a one-parameter family of Bloch-operators L^{ε} defined in (3.1) considered on the real Hilbert space $L^2_{\text{per}}([0,T])$. By standard results in Floquet theory, the spectrum of a given operator L^{ε} is discrete consisting of point eigenvalues and satisfy

$$\text{spec}_{L^2(\mathbb{R})}\left(\partial_x\mathcal{L}[u]\right)=\bigcup_{\varepsilon}\text{spec}(L^{\varepsilon})$$

and hence the $L^2(\mathbb{R})$ spectrum of the linearized operator $\partial_x\mathcal{L}[u]$ can be parameterized by the parameter ε. As a result, the above decomposition reduces the problem of determining the continuous spectrum of the operator $\partial_x\mathcal{L}[u]$ to that of determining the discrete spectrum of the one-parameter family of operators L^ε.

As we are interested in the modulational stability of the solution u, we begin our analysis by studying the null-space of the unperturbed operator $L_0 = \partial_x\mathcal{L}[u]$ acting on $L_{\rm per}([0,T])$. We will see that under certain nondegeneracy conditions L_0 has a two-dimensional kernel with a one-dimeisional Jordan chain. It follows that the origin is a T-periodic eigenvalue of L_0 with algebraic multiplicity three, and hence for $|\varepsilon| \ll 1$, considering L^ε as a small perturbation of L_0, there will be three eigenvalues bifurcating from the $\varepsilon = 0$ state. To determine modulational stability then, we will determine conditions which imply these bifurcating eigenvalues are confined to the imaginary axis.

To begin, we formalize the comments of the previous section concerning the structure of the generalized null-space of the unperturbed linearized operator $L_0 = \partial_x\mathcal{L}[u]$ by recalling the following lemma from [BrJ, BrJK].

Lemma 3.1 ([BrJ, BrJK]). *Suppose that $u(x; a_0, E_0, c_0)$ is a T-periodic solution of the traveling wave ordinary differential equation* (2.1)*, and that the Jacobian determinants T_E, $\{T, M\}_{a,E}$, and $\{T, M, P\}_{a,E,c}$ are nonzero at $(a_0, E_0, c_0) \in \mathcal{D}$. Then the functions*

$$\phi_0 = \{T, u\}_{a,E} \qquad \psi_0 = 1$$
$$\phi_1 = \{T, M\}_{a,E}\ u_x \qquad \psi_1 = \int_0^x \phi_2(s)ds$$
$$\phi_2 = \{u, T, M\}_{a,E,c} \qquad \psi_2 = \{T, M\}_{E,c} + \{T, M\}_{a,E}u$$

satisfy

$$L_0\phi_0 = L_0\phi_1 = 0 \qquad L_0^\dagger\psi_0 = L_0^\dagger\psi_2 = 0$$
$$L_0\phi_2 = -\phi_1 \qquad L_0^\dagger\psi_1 = \psi_2$$

In particular, if we further assume that $\{T, M\}_{a,E}$ and $\{T, M, P\}_{a,E,c}$ are nonzero at (a_0, E_0, c_0), then the functions $\{\phi_j\}_{j=1}^3$ forms a basis for the generalized null-eigenspace of L_0, and the functions $\{\psi_j\}_{j=1}^3$ forms a basis for the generalized null-eigenspace of $L_0^\dagger$. Furthermore, the orthogonality relations

$$\langle \psi_i, L_j\phi_k\rangle_{L^2_{\rm per}([0,T])} = 0$$

hold when $i + j + k = 0 \bmod(2)$.

Remark 3.2. *It should be clear that what is needed here is that the profile equation* (2.1) *should admit a "full family" of periodic traveling waves - an equation which is k^{th} order in space should admit a k parameter family of periodic traveling waves. This is what allows us to generate all of the necessary Bloch functions. This is true of gKdV along with a number of other nonlinear dispersive equations.*

Throughout the rest of our analysis, we will make the nondegeneracy assumption that the quantities T_E, $\{T, M\}_{a,E}$, and $\{T, M, P\}_{a,E,c}$ are nonzero (see the previous section).

Lemma 3.1 then implies, in essence, that the elements of the T-periodic kernel of the unperturbed operator L_0 are given by elements of the tangent space to the (two-dimensional) manifold of solutions *of fixed period and fixed wavespeed*, while the element of the first generalized kernel is given by a vector in the tangent space to the (three-dimensional) manifold of solutions *of fixed period* with no restrictions on wavespeed. It immediately follows that the origin is a T-periodic eigenvalue (corresponding to $\varepsilon = 0$, i.e. $\kappa = 0$) of L_0 of algebraic multiplicity three and geometric multiplicity two. Next, we vary κ in a neighborhood of zero to express the three eigenvalues bifurcating from the origin and consider the spectral problem

$$L^\varepsilon v(\varepsilon) = \mu(\varepsilon) v(\varepsilon)$$

for $|\varepsilon| \ll 1$, where we now make the additional assumption that the three branches of the function $\mu(\varepsilon)$ bifurcating from the $\mu(0) = 0$ state are distinct[3]. Our first goal is to show that the spectrum $\mu(\varepsilon)$ and hence the corresponding eigenfunctions $v(\varepsilon)$ are sufficiently smooth (C^1) in ε. The stronger result of analyticity was proved in [BrJ] using the Weierstrass-Preparation theorem and the Fredholm alternative. Here, we follow the methods from [JZ5] to offer an alternative proof which is more suitable for our methods.

Lemma 3.3. *Assuming the quantities $\{T, M\}_{a,E}$ and $\{T, M, P\}_{a,E,c}$ are nonzero, the eigenvalues $\mu_j(\varepsilon)$ of L^ε are C^1 functions of ε for $|\varepsilon| \ll 1$.*

Proof. To begin notice that since $\mu = 0$ is an isolated eigenvalue of L_0, the associated total right eigenprojection $R(0)$ and total left eigenprojection $L(0)$ perturb analytically in both ε (see [K]). It follows that we may find locally analytic right and left bases $\{v_j(\varepsilon)\}_{j=1}^3$ and $\{\tilde{v}_j(\varepsilon)\}_{j=1}^3$ of the associated total eigenspaces given by the range of the projections $R(\varepsilon)$ and $L(\varepsilon)$ such that $v_j(0) = \phi_j$ and $\tilde{v}_j(0) = \psi_j$. Further defining the vectors $V = (v_1, v_2, v_3)$ and $\tilde{V} = (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)^*$, where $*$ denotes the matrix adjoint, we may convert the infinite-dimensional perturbation problem for the operator L^ε to a 3×3 matrix perturbation problem for the matrix[4]

$$M_\varepsilon := \left\langle \tilde{V}^*(\varepsilon), L^\varepsilon V(\varepsilon) \right\rangle. \tag{3.2}$$

In particular, the eigenvalues of the matrix M_ε are coincide precisely with the eigenvalues $\mu_j(\varepsilon)$ of the operator L^ε lying in a neighborhood of $\mu = 0$, and the associated left and right eigenfunctions of L^ε are

$$f_j = V w_j \text{ and } \tilde{f}_j = \tilde{w}_j \tilde{V}^*$$

where w_j and $\tilde{w}_j$ are the associated right and left eigenvectors of M_ε, respectively. Thus, to demonstrate the desired smoothness of the spectrum of the operator L^ε near the origin

[3] In the case where two or more branches coincide to leading order, more delicate analysis is required than what is presented here. In particular, all asymptotic expansions must be continued to at least the next order in order to appropriately track all three branches.

[4] Throughout this work, the notation $\langle \cdot, \cdot \rangle$ will always denote the standard inner product on $L^2_{\text{per}}([0,T])$ given by $\langle g, h \rangle = \int_0^T gh\, dx$. Whether the inner product is for scalar or vector valued functions will be clear from context.

for $|\varepsilon| \ll 1$, we need only demonstrate that the three eigenvalues of the matrix M_ε have the desired smoothness properties, which is a seemingly much easier task.

Next, we expand the vectors

$$\begin{aligned} v_j(\varepsilon) &= \phi_j + \varepsilon q_j(\varepsilon) + \mathcal{O}(|\varepsilon|^2) \\ \tilde{v}_j(\varepsilon) &= \psi_j + \varepsilon \tilde{q}_j(\varepsilon) + \mathcal{O}(|\varepsilon|^2) \end{aligned} \tag{3.3}$$

and expand the matrix M_ε as

$$M_\varepsilon = M_0 + \varepsilon M_1 + \mathcal{O}(|\varepsilon|^2).$$

Notice that there is some flexibility in our choice of the functions q_j and $\tilde{q}_j$, a fact that we will exploit later. Now, however, our calculation does not depend on a particular choice of the expansions in (3.3). From Lemma 3.1 a straightforward computation shows that

$$M_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \langle \psi_1, L_0 \phi_2 \rangle \\ 0 & 0 & 0 \end{pmatrix} \tag{3.4}$$

where $\langle \psi_1, L_0 \phi_2 \rangle = \frac{1}{2}\{T, M\}_{a,E}\{T, M, P\}_{a,E,c}$ is nonzero by assumption reflecting the Jordan structure of the unperturbed operator L_0. By standard matrix perturbation theory, the spectrum of the matrix M_ε is C^1 in ε provided the entries $[M_1]_{1,2}$ and $[M_1]_{3,2}$ of the matrix M_1 are both zero. Using Lemma 3.1 again, it indeed follows that

$$\begin{aligned} [M_1]_{1,2} &= \langle \psi_0, L_1 \phi_1 + L_0 q_1 \rangle + \langle \tilde{q}_0, L_0 \phi_1 \rangle = 0 \\ [M_1]_{3,2} &= \langle \psi_2, L_1 \phi_1 + L_0 q_1 \rangle + \langle \tilde{q}_2, L_0 \phi_1 \rangle = 0 \end{aligned}$$

and hence the spectrum is $C^1(\mathbb{R})$ in the parameter ε near $\varepsilon = 0$ as claimed. □

As a result of Lemma 3.3, the associated (non-normalized) eigenfunctions bifurcating from the generalized null-space are C^1 in the parameter ε in a neighborhood of $\varepsilon = 0$. Moreover, our overall strategy is now clear: since the eigenvalues of M_ε correspond to the eigenvalues of the Bloch operator L^ε near the origin, we need only study the characteristic polynomial of the matrix M_ε near $\varepsilon = 0$ in order to understand the modulational stability of the underlying periodic wave. However, notice that by equation (3.4) the unperturbed matrix M_0 has a a non-trivial Jordan block, and hence the analysis of the bifurcating eigenvalues must be handled with care. In order to describe the breaking of the two-by-two Jordan block described in Lemma 3.1 for ε small, we rescale matrix M_ε in (3.2) as

$$\hat{M}_\varepsilon = \varepsilon^{-1} S(\varepsilon)^{-1} M_\varepsilon S(\varepsilon)$$

where

$$S(\varepsilon) = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}.$$

In particular, the matrix $\hat{M}_\varepsilon$ is an analytic matrix-valued function of ε, and the eigenvalues of $\hat{M}_\varepsilon$ are given by $\nu_j(\varepsilon) = \varepsilon^{-1}\mu_j(\varepsilon)$, where $\mu_j(\varepsilon)$ represents the eigenvalues of M_ε. Notice that the second coordinate of the vectors in $\mathbb{C}^{n+1}$ in the perturbation problem (3.2) corresponds to the coefficient of u_x to variations ψ in displacement. Thus, the above rescaling amounts to substituting for ψ the variable $|\varepsilon|\psi \sim \psi_x$ of the Whitham average system. Our next goal is to prove that the characteristic polynomial for the rescaled matrix M_ε agrees with that of the linearized dispersion relation $\hat{\Delta}(\mu,\kappa)$ corresponding to the homogenized system (2.12), i.e. we want to prove that

$$\det\left(\varepsilon\hat{M}_\varepsilon - \mu\left\langle S(\varepsilon)^{-1}\tilde{V}_\varepsilon^*, V_\varepsilon S(\varepsilon)\right\rangle\right) = \hat{\Delta}(\mu,\kappa) + \mathcal{O}(|\mu|^4 + |\kappa|^4)$$

for some nonzero constant C. To this end, we begin by determining the required variations in the vectors $\tilde{V}_\varepsilon$ and V_ε near $\varepsilon = 0$ contribute to leading order in the above equation. We begin by studying the structure of the matrix $\hat{M}_\varepsilon$.

Lemma 3.4. *The matrix rescaled $\hat{M}_\varepsilon$ can be expanded as*

$$\hat{M}_\varepsilon = \varepsilon^{-1}\left\langle S^{-1}(\varepsilon)\begin{pmatrix}\psi_0+\tilde{q}_0\\ \psi_1\\ \psi_2+\varepsilon\tilde{q}_2\end{pmatrix}, L^\varepsilon\left(\phi_0,\phi_1+\varepsilon q_1,\phi_2\right)S(\varepsilon)\right\rangle + o(1)$$

in a neighborhood of $\varepsilon = 0$. In particular, the only first order variations of the vectors V_ε and $\tilde{V}_\varepsilon$ which contribute to leading order are $\tilde{q}_0$, $\tilde{q}_2$ and q_1.

Proof. The idea is to undo the rescaling and find which entries of the unscaled matrix M_ε contribute to leading order. To begin, we expand the non-rescaled matrix M_ε from (3.2) as

$$M_\varepsilon = M_0 + \varepsilon M_1 + \varepsilon^2 M_2 + \mathcal{O}(|\varepsilon|^3)$$

and notice that M_0 was computed in (3.4) and was shown to be nilpotent but nonzero, possessing a nontrivial associated Jordan chain of height two. Using Lemma 3.1 a straightforward computation shows the matrix M_1 can be expressed as

$$\begin{aligned}M_1 &= \begin{pmatrix}\langle\psi_0, L_1\phi_0 + L_0 q_0\rangle & 0 & \langle\psi_0, L_1\phi_2\rangle + \langle\tilde{q}_0, L_0\phi_2\rangle\\ * & \langle\psi_1, L_1\phi_1 + L_0 q_1\rangle & *\\ \langle\psi_2, L_1\phi_0 + L_0 q_0\rangle & 0 & \langle\psi_2, L_1\phi_2\rangle + \langle\tilde{q}_2, L_0\phi_2\rangle\end{pmatrix}\\ &= \begin{pmatrix}\langle\psi_0, L_1\phi_0\rangle & 0 & \langle\psi_0, L_1\phi_2\rangle + \langle\tilde{q}_0, L_0\phi_2\rangle\\ * & \langle\psi_1, L_1\phi_1 + L_0 q_1\rangle & *\\ \langle\psi_2, L_1\phi_0\rangle & 0 & \langle\psi_2, L_1\phi_2\rangle + \langle\tilde{q}_2, L_0\phi_2\rangle\end{pmatrix}\end{aligned}$$

where the $*$ terms are not necessary as they contribute to higher order terms in the rescaled matrix $\hat{M}_\varepsilon$. Similarly, the relevant entries of the matrix M_2 are given by

$$M_2 = \begin{pmatrix} * & \langle\psi_0, L_2\phi_1 + L_1 q_1\rangle + \langle\tilde{q}_0, L_1\phi_1 + L_0 q_1\rangle & *\\ * & * & *\\ * & \langle\psi_2, L_2\phi_1 + L_1 q_1\rangle + \langle\tilde{q}_2, L_1\phi_1 + L_0 q_1\rangle & *\end{pmatrix}.$$

Therefore, it follows the relevant entries of the matrix M_ε up to $\mathcal{O}(|\varepsilon|^2)$ can be evaluated using only the variations q_1 and $\tilde{q}_2$, as claimed. ☐

The main point of the above lemma is that in order to compute to projection of the operator L^ε onto the eigenspace bifurcating null-space at $\varepsilon = 0$ to leading order, we only need to consider the variations in the bottom of the left and right Jordan chains; all other variations contribute to terms of higher order. Our next lemma shows that the variation in the ψ_0 direction is also needed to compute the corresponding projection of the identity to leading order.

Lemma 3.5. *Define the matrix* $\tilde{I}_\varepsilon := \left\langle S(\varepsilon)^{-1}\tilde{V}_\varepsilon^*, \tilde{V}_\varepsilon S(\varepsilon)\right\rangle$. *Then* $\tilde{I}_\varepsilon$ *can be expanded near* $\varepsilon = 0$ *as*

$$\tilde{I}_\varepsilon = \left\langle S^{-1}(\varepsilon)\begin{pmatrix} \psi_0 + \varepsilon\tilde{q}_0 \\ \psi_1 \\ \psi_2 + \varepsilon\tilde{q}_2 \end{pmatrix}, (\phi_0, \phi_1 + \varepsilon q_1, \phi_2)\, S(\varepsilon)\right\rangle + o(1).$$

Proof. As in the proof of Lemma 3.4, we undo the rescaling and find the terms that contribute to leading order. To begin, we expand the matrix $I_\varepsilon := \left\langle \tilde{V}_\varepsilon^*, V_\varepsilon\right\rangle$ as

$$I_\varepsilon = I_0 + \varepsilon I_1 + \mathcal{O}(|\varepsilon|^2)$$

Using the fact that $\langle\psi_i, \phi_j\rangle = 0$ if $i \neq j$, it follows that

$$I_0 = \begin{pmatrix} \langle\psi_0, \phi_0\rangle & 0 & 0 \\ 0 & \langle\psi_1, \phi_1\rangle & 0 \\ 0 & 0 & \langle\psi_2, \phi_2\rangle \end{pmatrix}$$

and

$$I_1 = \begin{pmatrix} * & \langle\psi_0, q_1\rangle + \langle\tilde{q}_0, \phi_1\rangle & * \\ * & * & * \\ * & \langle\psi_2, q_1\rangle + \langle\tilde{q}_2, \phi_1\rangle & * \end{pmatrix}$$

from which the lemma follows by rescaling. □

With the above preparations, we can now project the operator $L^\varepsilon - \mu(\varepsilon)$ onto the total eigenspace bifurcating from the origin for small $|\varepsilon| \ll 1$. Our claim is that the projected and rescaled matrix agrees with the symbol of the linearized Whitham averaged system, up to a similarity transformation. To show this, we begin by showing the characteristic polynomial of the corresponding rescaled matrix agrees with the linearized dispersion relation $\hat{\Delta}(\mu, \kappa)$ corresponding to the linearized Whitham averaged system. Once this is established, the fact that the bifurcating eigenvalues are distinct will imply the desired similarity.

In order to compute the characteristic polynomial of the matrix projection of the operator $L^\varepsilon - \mu(\varepsilon)$, we must now make a few specific choices as to the variations in the vectors $\tilde{V}_\varepsilon$ and V_ε. As noted in the proof of Lemma 3.3, there is quite a bit of flexibility in our choice of the expansion (3.3). As a naive choice, we could require that $v_j(\varepsilon)$ in (3.3) to be an eigenfunction of L^ε for small ε. For example, we could choose the q_j to satisfy

$$L_0 q_j = (\lambda_1 - L_1)\phi_j,$$

where λ_1 is the corresponding eigenvalue bifurcating from the origin. Similarly we could choose $\tilde{q}_j$ to satisfy

$$L_0^\dagger \tilde{q}_j = (\lambda_1^* - L_1)\psi_j,$$

where $*$ denotes complex conjugation and, indeed, we do make this choice for the variation $\tilde{q}_0$. However, these choices for q_1 and $\tilde{q}_2$ (the variations at the bottom of the Jordan chains) lead to very cumbersome calculations; see section 4 of [BrJ] where these choices were made and a Bloch-based modulational stability analysis was attempted. Instead, here we make the observation that we only need the functions $v_j(\varepsilon)$ in (3.3) to provide a *basis*, not necessarily an eigenbasis, for the near zero eigenspace of the operator L^ε. To illustrate this, instead of choosing the variation in the ϕ_1 direction to satisfy an eigenvalue equation we notice there are two distinct eigenvalues with expansions

$$\mu_1(\varepsilon) = \varepsilon\lambda_1 + o(|\varepsilon|),$$
$$\tilde{\mu}_1(\varepsilon) = \varepsilon\tilde{\lambda}_1 + o(|\varepsilon|)$$

corresponding the eigenfunctions $\phi_1 + \varepsilon g + o(|\varepsilon|)$ and $\phi_1 + \varepsilon\hat{g} + o(|\varepsilon|)$. In particular, it follows that the functions g and $\hat{g}$ satisfy the equations

$$L_0 g = (\lambda_1 - L_1)\phi_1,$$
$$L_0 \hat{g} = (\hat{\lambda}_1 - L_1)\phi_1.$$

Defining $q_1 = \left(\hat{\lambda}_1 - \lambda_1\right)^{-1}\left(\hat{\lambda}_1 g - \lambda_1 \hat{g}\right)$ then, it follows that the function q_1 to satisfies

$$\begin{aligned} L_0 q_1 &= \left(\hat{\lambda}_1 - \lambda_1\right)^{-1}\left(\hat{\lambda}_1(\lambda_1 - L_1)\phi_1 - \lambda_1(\hat{\lambda}_1 - L_1)\phi_1\right) \\ &= -L_1\phi_1 \end{aligned}$$

and hence may be chosen to satisfy $q_1 \perp \operatorname{span}\{\psi_0, \psi_1\}$. To find a closed form expression for q_1, notice that

$$L_0\left(x u_x\right) = 2u_{xxx} = -L_1 u_x.$$

Moreover, a direct calculation shows that the function u_E satisfies $\mathcal{L}[u]u_E = 0$ and the function

$$\tilde{\phi} = -\{T, M\}_{a,E}\left(x u_x + \frac{T}{T_E} u_E\right)$$

is T-periodic. In particular, $L_0\tilde{\phi} = -L_1\phi_1$ and $\tilde{\phi} \perp \operatorname{span}\{\psi_0, \psi_1\}$ and hence we may choose $q_1 = \tilde{\phi}$. Note there are many other choices for q_1 that are possible; our choice is made to simplify the forthcoming calculations. Similarly, since ψ_2 is at the bottom of the Jordan chain of the null-space of $L_0^\dagger$, we may choose $\tilde{q}_2$ to satisfy the equation

$$L_0^\dagger \tilde{q}_2 = -L_1^\dagger \psi_2.$$

For $\tilde{q}_2$, unlike the variation in ϕ_1, we do not need a closed form expression for $\tilde{q}_2$; the above defining relation will be sufficient for our purposes.

With the above choices, all the necessary inner products described in Lemmas 3.4 and 3.5 may be evaluated explicitly. Indeed, straightforward computations show that

$$M_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\{T,M\}_{a,E}\{T,M,P\}_{a,E,c} \\ 0 & 0 & 0 \end{pmatrix}$$

$$M_1 = \begin{pmatrix} T_E T & * & 0 \\ * & 0 & * \\ T(T_E\{T,M\}_{E,c} + T a\{T,M\}_{a,E}) & * & 0 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} * & T\{T,K\}_{a,E} & * \\ * & * & * \\ * & T\{T,M\}_{E,c}\{T,K\}_{a,E} + \frac{T\{T,M\}_{a,E}}{T_E}\left(T_a\{T,K\}_{a,E} - T\{T,M\}_{a,E}\right) & * \end{pmatrix}.$$

and

$$I_0 = \begin{pmatrix} \{T,M\}_{a,E} & 0 & 0 \\ 0 & -\frac{1}{2}\{T,M\}_{a,E}\{T,M,P\}_{a,E,c} & 0 \\ 0 & 0 & \frac{1}{2}\{T,M\}_{a,E}\{T,M,P\}_{a,E,c} \end{pmatrix}$$

$$I_1 = \begin{pmatrix} * & -\{T,M\}_{E,c}T - \{T,M\}_{a,E}M & * \\ * & * & * \\ * & 2\{T,M\}_{a,E}\{K,T,M\}_{a,E,c} - \{T,M\}_{E,c}^2 T - 2\{T,M\}_{E,c}\{T,M\}_{a,E}M - \{T,M\}_{a,E}^2 P & * \end{pmatrix}.$$

We can thus explicitly compute the rescaled matrices $\hat{M}_\varepsilon$ and $\tilde{I}_\varepsilon$ in terms of the underlying solution u, which yields the following theorem.

Theorem 2. *Let* $(a_0, E_0, c_0) \in \mathcal{D}$ *and assume the matrices* $\frac{\partial(M\omega, P\omega, \omega)}{\partial(\dot{u})}$ *and* $\frac{\partial(\dot{u})}{\partial(a,E,c)}$ *are invertible at* (a_0, E_0, c_0)*. Then the linearized dispersion relation* $\hat{\Delta}(\mu,\kappa)$ *in* (2.13) *satisfies*

$$\det\left(\varepsilon \hat{M}_\varepsilon - \mu\left\langle S(\varepsilon)^{-1}\tilde{V}_\varepsilon^*, V_\varepsilon S(\varepsilon)\right\rangle\right) = C\hat{\Delta}(\mu,\kappa) + \mathcal{O}(|\mu|^4 + |\kappa|^4)$$

for some constant $C \neq 0$*. That is, up to a constant the linearized dispersion relation for the homogenized system* (2.12) *accurately describes the low-frequency behavior of the spectrum of the Bloch-operator* L_ξ*.*

Proof. A straightforward computation using the above identities implies

$$\det\left(\varepsilon \hat{M}_\varepsilon - \mu\left\langle S(\varepsilon)^{-1}\tilde{V}_\varepsilon^*, V_\varepsilon S(\varepsilon)\right\rangle\right) = \frac{T^3\{T,M\}_{a,E}^3}{2T_E}\left(\{T,M\}_{E,a} + T_a^2 - 2T_c T_E\right) + C\hat{\Delta}(\mu,\kappa) + \mathcal{O}(|\mu|^4 + |\kappa|^4)$$

for some nonzero constant $C = C(a,E,c)$. Moreover, using the identity

$$2T_c = P_E = M_a = -\frac{\sqrt{2}}{4}\oint_\Gamma \frac{u^2\,du}{(E - V(u;a,c))^{3/2}},$$

which immediately follow from the integral formulas (2.3)-(2.5), along with the fact that $T_a = M_E$ by (2.7), it follows that

$$T_a^2 - 2T_cT_E = T_aM_E - T_EM_a = \{T, M\}_{a,E} = -\{T, M\}_{E,a}$$

and hence

$$\det\left(\varepsilon\hat{M}_\varepsilon - \mu\left\langle S(\varepsilon)^{-1}\tilde{V}_\varepsilon^*, V_\varepsilon S(\varepsilon)\right\rangle\right) = C\hat{\Delta}(\mu,\kappa) + \mathcal{O}(|\mu|^4 + |\kappa|^4)$$

as claimed. □

Theorem 2 provides a rigorous verification at a linearized level of the Whitham modulation equations for the gKdV equations. While this has recently been established in [JZ1], the important observation here is that the verification was independent of the restrictive Evans function techniques. Thus, the above computation can be used as a blueprint for how to rigorously justify Whitham expansions in more complicated settings where the Evans function framework is not available. As a consequence of our assumption that the eigenvalues of M_ε be distinct for $0 < |\varepsilon| \ll 1$, it follows that there must exist a similarity transformation between the matrix

$$\partial_\varepsilon\left(\varepsilon\hat{M}_\varepsilon - \mu\left\langle S(\varepsilon)^{-1}\tilde{V}_\varepsilon^*, V_\varepsilon S(\varepsilon)\right\rangle\right)\Big|_{\varepsilon=0}$$

and the matrix arising from the linearized Whitham averaged system (2.13). Thus, the variations predicted by Whitham to control the long-wavelength stability of a periodic traveling wave solution of the gKdV are indeed the variations needed at the level of the linearized Bloch-expansion.

4 Conclusions

In this paper we have considered the spectral stability of a periodic traveling wave of the gKdV equation to long-wavelength perturbations. Recently, this notion of stability has been the focus of much work in the context of viscous systems of conservation laws [OZ1, OZ3, OZ4, Se1, JZ4, JZ5] and nonlinear dispersive equations [BrJ, BrJK, J2, JZ]. While much of this work has utilized the now familiar and powerful Evans function techniques, our approach of using a direct Bloch-decomposition of the linearized operator serves to provide an elementary method in the one-dimensional setting considered here as well as a robust method which applies in the more complicated setting of multi-periodic structures. As such, our hope is that the simple and straightforward analysis in this paper will be used as a blueprint for how to justify the Whitham modulation equations (at a linearized level) in settings which do not admit a readily computable Evans function. In future work, we hope to apply this method to the long-wavelength stability of doubly periodic traveling waves of viscous systems of conservation laws.

As a future direction of investigation, we note that Theorem 2 and the assumption that the branches of spectrum bifurcating from the origin are distinct suggests a possibly more algorithmic approach to justifying (at a linearized level) the Whitham modulation equations via the above Bloch-expansion methods which furthermore does not rely on comparing the linearized dispersion relations as in Theorem 2; it should be possible to justify the Whitham expansion by direct comparison of inner products and showing the mentioned similarity directly. As a first step, we suggest beginning this line of investigation by studying the gKdV equation as in the current paper and [JZ1] and expressing the full linearized Whitham averaged system in terms of inner products. Demonstrating the desired similarity in this way, while not completely necessary, could substantially simplify the amount of work required to justify the linearized Whitham averaged equations by not requiring an initial justification at the spectral level (a possibly daunting task in more general situations).

Finally, we wish to stress once more that the analysis presented in this paper justifies the Whitham equations at a linearized level. In particular, the methods provide a method to prove that by averaging and linearizing the Whitham modulation equations for the gKdV equation accurately describes the spectrum of the linearized operator in a neighborhood of the origin, i.e. correctly predicts the spectral stability of the underlying periodic wave to long wavelength perturbations. A justification at the nonlinear level is well beyond the scope of our analysis and presents a formidable and interesting open problem.

References

[B] T. B. Benjamin, *The stability of solitary waves.* Proc. Roy. Soc. (London) Ser. A, 328 (1972).

[BD] N. Bottman and B. Deconinck, *KdV cnoidal waves are linearly stable*, DCDS-A 25 (2009).

[BrJ] J. C. Bronski and M. Johnson, *The modulational instability for a generalized Korteweg-de Vries equation*, Arch. Ration. Mech. Anal., 197 (2010), no. 2.

[BrJK] J. C. Bronski, M. Johnson, and T. Kapitula, *An index theorm for the stability of periodic traveling waves of KdV type*, preprint.

[BM] E. Barthelemy and H. Michallet, *Experimental study of large interfacial solitary waves*, Journal of Fluid Mechanics, 336 (1998).

[BaMo] J. F. Bampi and A. Morro, *Korteweg-de Vries equation and nonlinear waves*, Lettere Al Nuovo Cimento, 26 (1979), no. 2.

[Bo] J. L. Bona, *On the stability theory of solitary waves*, Proc. Roy. Soc. London Ser. A, 344 (1975).

[BO] T. L. Burch and A. R. Osborne, *Internal Solitons and the Andaman sea*, Science, 208 (1940), no. 4443.

[DK] B. Deconinck and T. Kapitula, *On the orbital (in)stability of spatially periodic stationary solutions of generalized Korteweg-de Vries equations*, preprint.

[FFM] H. Flaschka, M. G. Forest, and D. W. McLaughlin, *Multiphase averaging and the inverse spectral solution of the Korteweg-de Vries equation*, Comm. Pure Appl. Math., 33 (1980).

[GH] Th. Galley and M. Hărăguş, *Stability of small periodic waves for the nonlinear Schrdinger equation*, J. Diff. Equations, 234 (2007).

[Ha] M. Hărăguş, *Stability of periodic waves for the generalized BBM equation*, Rev. Roumaine Maths. Pures Appl., 53 (2008).

[HK] M. Hărăguş and T. Kapitula, *On the spectra of periodic waves for infinite-dimensional Hamiltonian systems*, Physica D, 237 (2008), no. 20.

[HaLS] M. Hărăguş, E. Lombardi, and A. Scheel, *Stability of wave trains in the Kawahara equation*, J. Math. Fluid Mech., 8 (2006).

[J1] M. Johnson, *Nonlinear Stability of Periodic Traveling Wave Solutions of the Generalized Korteweg-de Vries Equation*, SIMA, 41 (2009), no. 5.

[J2] M. Johnson, *On the Stability of Periodic Solutions of the Generalized Benjamin-Bona-Mahony Equation*, Physica D, to appear.

[J3] M. Johnson, *The Transverse Instability of Periodic Waves in Zakharov-Kuznetsov Type Equations*, Studies in Applied Mathematics, 124 (2010), no. 4.

[JZ] M. Johnson and K. Zumbrun, *Transverse Instability of Periodic Traveling Waves in the Generalized Kadomtsev-Petviashvili Equation*, submitted, (2009).

[JZ1] M. Johnson and K. Zumbrun, *Rigorous Justification of the Whitham Modulation Equations for the Generalized Korteweg-de Vries Equation*, Studies in Applied Mathematics, to appear, DOI:10.1111/j.1467-9590.2010.00482.x, (2010).

[JZ4] M. Johnson and K. Zumbrun, *Nonlinear stability of periodic traveling wave solutions of systems of conservation laws in dimensions one and two*, submitted, (2009).

[JZ5] M. Johnson and K. Zumbrun, *Nonlinear stability of periodic traveling wave solutions of systems of viscous conservation laws in the generic case*, Journal of Differential Equations, to appear, DOI:10.1016/j.jde.2010.04.015, (2010).

[KD] B. C. Kalita and R. Das, *Modified Korteweg de Vries (mkdv) and Korteweg-de Vries (kdv) solitons in a warm plasma with negative ions and electrons' drift motion*, Journal of the Physical Socity of Japan, 71 (2002), no. 21.

[LL] P. D. Lax and D. C. Levermore, *The small dispersion limit of the Korteweg-de Vries equation. I,II,III*, Comm. Pure Appl. Math., 36 (1983), no. 3.

[M] A. A. Mamun, *Nonlinear propagation of ion-acoustic waves in a hot magnetized plasma with vortex-like electron distribution*, Physics of Plasmas, 5 (1998), no. 1.

[MS] A. Mushtaq and H. A. Shah, *Study of non-maxwellian trapped electrons by using generalized (r,q) distribution function and their effects on the dynamics of ion acoustic solitary wave*, Physics of Plasmas, 13 (2006).

[K] T. Kato, *Perturbation theory for linear operators*, Springer–Verlag, Berlin Heidelberg (1985).

[KdV] D. J. Korteweg and G. de Vries, *On the change of form of long waves advancing in a rectangular channel and on a new type of stationary waves*, Philosophical Magazine, 36 (1895).

[PW] R. L. Pego and M. I. Weinstein, *Eigenvalues and instabilities of solitary waves*, Philos. Trans. Roy. Soc. London Ser. A., 340 (1992).

[OZ1] M. Oh and K. Zumbrun, *Stability of periodic solutions of viscous conservation laws with viscosity- 1. Analysis of the Evans function*, Arch. Ration. Mech. Anal. 166 (2003), no. 2, 99–166.

[OZ3] M. Oh, and K. Zumbrun, *Low-frequency stability analysis of periodic traveling-wave solutions of viscous conservation laws in several dimensions*, Journal for Analysis and its Applications, 25 (2006), 1–21.

[OZ4] M. Oh, and K. Zumbrun, *Stability and asymptotic behavior of traveling-wave solutions of viscous conservation laws in several dimensions*, to appear, Arch. Ration. Mech. Anal.

[Ro] G. B. Rowlands, *On the stability of solutions of the non-linear Schrödinger equation*, J. Inst. Maths. Applics., 13 (1974): 367-377.

[Se1] D. Serre, *Spectral stability of periodic solutions of viscous conservation laws: Large wavelength analysis*, Comm. Partial Differential Equations 30 (2005), no. 1-3, 259–282.

[SH] H. Segur and J. Hammack, *Soliton models of long internal waves*, Journal of Fluid Mechanics, 118 (1982).

[W] G. B. Whitham, *Linear and Nonlinear Waves*, Pure and Applied Mathematics (New York), John Wiley & Sons Inc., New York, 1999. Reprint of the 1974 original, A Wiley-Interscience Publication.