Rigorous Justification of the Whitham Modulation Equations for the Generalized Korteweg-de Vries Equation *

Mathew A. Johnson† Kevin Zumbrun‡

March 24, 2010

Keywords: Generalized Korteweg-de Vries equation; Periodic waves; Modulational stability; Whitham theory; WKB.

Abstract

In this paper, we consider the spectral stability of spatially periodic traveling wave solutions of the generalized Korteweg-de Vries equation to long-wavelength perturbations. Specifically, we extend the work of Bronski and Johnson by demonstrating that the homogenized system describing the mean behavior of a slow modulation (WKB) approximation of the solution correctly describes the linearized dispersion relation near zero frequency of the linearized equations about the background periodic wave. The latter has been shown by rigorous Evans function techniques to control the spectral stability near the origin, i.e. stability to slow modulations of the underlying solution. In particular, through our derivation of the WKB approximation we generalize the modulation expansion of Whitham for the KdV to a more general class of equations which admit periodic waves with nonzero mean. As a consequence, we will show that, assuming a particular non-degeneracy condition, spectral stability near the origin is equivalent with the local well-posedness of the Whitham system.

1 Introduction

In the area of nonlinear dispersive waves the question of stability is of fundamental importance as it determines those solutions which are most likely to be observed in physical applications. In this paper, we consider the stability of the spatially periodic traveling wave solutions of the generalized Korteweg-de Vries (gKdV) equation

$$u_t = u_{xxx} + f(u)_x, \tag{1.1}$$

*This is an electronic version of an Article published in *Studies in Applied Mathematics*. Full citation: Studies in Applied Mathematics, DOI: 10.1111/j.1467-9590.2010.00482.x, (2010).

†Indiana University, Bloomington, IN 47405; matjohn@indiana.edu: Research of M.J. was partially supported by an NSF Postdoctoral Fellowship under NSF grant DMS-0902192.

‡Indiana University, Bloomington, IN 47405; kzumbrun@indiana.edu: Research of K.Z. was partially supported under NSF grants no. DMS-0300487 and DMS-0801745.

where u is a scalar, $x, t \in \mathbb{R}$ and f is a suitable nonlinearity. Such equations arise in a variety of applications. For example, the case $f(u) = u^2$ corresponds to the well known Korteweg-de Vries (KdV) equation which arises as a canonical model of weakly dispersive nonlinear wave propagation [17] [22]. Moreover, the cases $f(u) = \beta u^3$, $\beta = \pm 1$, correspond to the modified KdV equation which arises as a model for large amplitude internal waves in a density stratified medium, as well as a model for Fermi–Pasta-Ulam lattices with bistable nonlinearity [1] [4]. In each of these two cases, the corresponding PDE can be realized as a compatibility condition for a particular Lax pair and hence the corresponding Cauchy problem can (in principle) be completely solved via the famous inverse scattering transform[1]. However, there are a variety of applications in which equations of form (1.1) arise which are not completely integrable and hence the inverse scattering transform can not be applied. For example, in plasma physics equations of the form (1.1) arise with a wide variety of power-law nonlinearities depending on particular physical considerations [16] [18] [19]. Thus, in order to accommodate as many applications as possible, the methods employed in this paper will not rely on complete integrability of the PDE (1.1). Instead, we will make use of the integrability of the ordinary differential equation governing the traveling wave profiles. This ODE is always Hamiltonian, regardless of the integrability of the corresponding PDE.

The stability of such solutions has received much attention recently. For example, see [5], [6], [7], [8], [9], [10], [12], [13]. Here, we are interested in the spectral stability to long wavelength perturbations, i.e. to slow modulations of the underlying wave. There is a well developed (formal) physical theory for dealing with such problems which is known as Whitham modulation theory [27] [28]. This formal theory proceeds by rescaling the governing PDE via the change of variables $(x, t) \mapsto (\varepsilon x, \varepsilon t)$, then uses a WKB approximation of the solution and looks for a homogenized system which describes the mean behavior of the resulting approximation. Heuristically then, one may expect that a necessary condition for the stability of such solutions is the hyperbolicity– i.e., local well-posedness– of the resulting first order system of partial differential equations. In order to make this intuition rigorous, one must study in detail the spectrum of the linearized operator about a periodic solution in a neighborhood of the origin in the spectral plane, and compare the resulting low-frequency stability region to that predicted by hyperbolicity of the formal Whitham expansion.

The first part of this analysis was recently conducted by Bronski and Johnson [6]. There, the authors studied the spectral stability of spatially periodic traveling wave solutions to (1.1) to long-wavelength perturbations by using periodic Evans function techniques, i.e. Floquet theory. In particular, an index was derived whose sign, assuming a particular non-degeneracy condition holds, determines the local structure of the spectrum near the origin in the spectral plane. This index arises as the discriminant of a polynomial which encodes the leading order behavior of the Evans function near the origin for such perturbations: that is, it describes the linearized dispersion relation near zero frequency of the corresponding

[1] More precisely, the Cauchy problem for equation (1.1) can be solved via the inverse scattering transform if and only if f is a cubic polynomial.

linearized equations about the background periodic wave. Notice that the origin in the spectral plane is distinguished by the fact that the tangent space to the manifold of traveling wave profiles can be explicitly computed, and the generalized null space of the corresponding linearized operator is spanned by a particular basis of this tangent space (again, assuming a particular non-degeneracy condition is met). In particular, Bronski and Johnson showed that if one assumes the conserved quantities of the PDE (1.1) provide good local coordinates for near by periodic traveling waves, then the generalized periodic null space of the linearized operator is generated by taking specific variations in the parameters defining the periodic traveling waves. As a result, the linearized dispersion relation near zero frequency can be explicitly computed in terms of Jacobians of various maps from the traveling wave parameters to the conserved quantities of the PDE flow. These Jacobians were then shown to be explicitly computable in terms of moments of the underlying wave in the case of power-law nonlinearity $f(u) = u^{p+1}$, $p \in \mathbb{N}$ (see [7]).

The purpose of this paper is to carry out the second part of the program, connecting the formal Whitham procedure with the rigorous results of [6]. To motivate our approach, recall that in [28] Whitham caries out the formal modulation approximation for the cnoidal wave solutions of the KdV equation, in which everything can be evaluated explicitly in terms of elliptic integrals. As a first step in his analysis, Whitham introduces a periodic potential function ϕ with the requirement $u = \phi_x$ where u is a given solution of the KdV equation. In effect, the introduction of this potential allows for a Lagrangian formulation of the KdV which can be analyzed via the modulation theory presented in [28]. However, notice that by requiring the solution u to be of divergence form one is forcing that the solution have mean zero over one period. This of course presents no problem in the case of the KdV equation since all solutions can be made mean zero by Galilean invariance, but it does pose a serious problem when trying to extend Whitham's method to more general equations of form (1.1) which admit periodic waves of non-zero mean; for example, the approximation would not be valid for the well known elliptic function solutions of either the focusing or defocusing modified KdV equation.

In order to consider more general equations of the form (1.1), it becomes imperative to find a way to side step this restriction to mean zero waves. To this end we follow the novel approach suggested by Serre [26], which is to work with the original variable u and augment the resulting WKB system with an additional conservation law associated with the Hamiltonian structure/translation invariance of the gKdV; see also the recent work of Zumbrun and Oh [20] in which this approach was used in the viscous conservation law setting. As we will see, this approach not only works in the KdV case originally considered by Whitham, but also for the general Hamiltonian gKdV. In particular, we will see that the linearized dispersion relation predicted by the linearized Whitham system correctly predicts (assuming particular non-degeneracy conditions are met) the modulational stability of the given periodic wave. To this end, we compare the linearized dispersion relation coming from the modulation approximation to that derived by Bronski and Johnson using Evans function techniques. Differently from in the works of Serre and Zumbrun and Oh [26, 20], however, we show the equivalence by an essentially different method using direct computation and

identities derived in [6] as opposed to direct linear-algebraic comparisons. The remarkable fact that the resulting homogonized system can be explicitly computed in terms of the conserved quantities of the PDE flow and the parameters defining the periodic traveling waves seems to be very special to the case of the gKdV equation (1.1) in which the ODE defining the traveling wave profiles is completely integrable. We expect that the approach proposed by Serre to show this equivalence should work in this case as well, by using the fact that additional conservation laws gives rise to additional first order variations, but the direct method employed here seems simpler in the present context.

The results in this paper therefore extend the original work of Whitham concerning the modulational stability of cnoidal waves of the KdV to more general periodic waves of equations of form (1.1), and also rigorously validate the predictions of the Whitham expansion as regards linearized stability about a periodic wave. However, as pointed out in [20, 21], this is only the begining of the story. The importance of the connection between Whitham expansion and linearized stability comes rather from the predictive power of the formal expansion in deriving more detailed information about behavior of solutions. See also the earlier work of Schneider [23, 24, 25] for reaction–diffusion and other equations. Thus, as in the related contexts [23, 24, 25, 20, 21], we expect the validation of this connection to lead to further important results about these and other systems.

The outline of this paper is as follows. In section 2 we recall the basic properties of the periodic traveling wave solutions of (1.1) described in [6] and [13]. In particular, we discuss a parametrization of such solutions which will be utilized throughout the present work. In section 3, we outline the results of Bronski and Johnson concerning the low frequency asymptotics of the periodic Evans function and describe the connection to the modulational stability of the underlying periodic wave. In section 4, we carry out the slow modulation (WKB) approximation to derive the Whitham system for the gKdV equation. In particular, we connect this formal homogenization procedure to the low frequency analysis of Bronski and Johnson. We then close with a discussion and open problems.

2 Periodic Solutions of the gKdV

Throughout this paper, we are concerned with the periodic traveling wave solutions of the gKdV equation. To begin then, we recall the basic properties of the periodic traveling wave solutions of (1.1). For more details, see [6], [7], or [13].

Such solutions are stationary solutions of (1.1) in a moving coordinate frame of the form $x + ct$ and whose profiles satisfy the traveling wave ordinary differential equation

$$u_{xxx} + f(u)_x - cu_x = 0. \tag{2.1}$$

This profile equation is Hamiltonian and hence can be reduced through two integrations to the nonlinear oscillator equation

$$\frac{u_x^2}{2} = E + au + \frac{c}{2}u^2 - F(u), \tag{2.2}$$

where $F' = f$, $F(0) = 0$, and a and E are constants of integration. Thus, the existence of periodic orbits of (2.1) can verified through simple phase plane analysis: a necessary and sufficient condition is for the effective potential energy $V(u; a, c) = F(u) - au - \frac{c}{2}u^2$ to have a local minimum. It follows that the periodic solutions of (2.1) form, up to translation[2], a three parameter family of traveling wave solutions of (1.1) which we can parameterize by the constants a, E, and c. In particular, on open sets in $\mathbb{R}^3 = (a, E, c)$ the solution to (2.2) is periodic: the boundary of these open sets corresponds to the equilibrium solutions and solitary waves (homoclinic/heteroclinic orbits).

In addition, we make the assumption that there exist simple roots $u_\pm$ of the equation $E = V(u; a, c)$ which satisfy $u_- < u_+$, and that $V(u; a, c) < E$ for $u \in (u_-, u_+)$. As a consequence, the roots $u_\pm$ are C^1 functions of the traveling wave parameters (a, E, c) and, without loss of generality, we can assume that $u(0) = u_-$. It follows that the period of the corresponding periodic solution of (2.1) can be expressed by the formula

$$T = T(a, E, c) = \sqrt{2} \int_{u_-}^{u_+} \frac{du}{\sqrt{E - V(u; a, c)}} = \frac{\sqrt{2}}{2} \oint_\Gamma \frac{du}{\sqrt{E - V(u; a, c)}}, \tag{2.3}$$

where integration over Γ represents a complete integration from u_- to u_+, and then back to u_- again. Notice however that the branch of the square root must be chosen appropriately in each direction. Alternatively, the contour Γ can be interpreted as a loop (Jordan curve) in the complex plane which encloses a bounded set containing both u_- and u_+. By a standard procedure, the above integral can be regularized at the square root branch points and hence represents a C^1 function of the traveling wave parameters. See [6] for details.

Notice that in general, the gKdV equation admits three conserved quantities[3] which can be represented as

$$M(a, E, c) = \int_0^T u(x)dx = \frac{\sqrt{2}}{2} \oint_\Gamma \frac{u\ du}{\sqrt{E - V(u; a, c)}} \tag{2.4}$$

$$P(a, E, c) = \int_0^T u(x)^2 dx = \frac{\sqrt{2}}{2} \oint_\Gamma \frac{u^2\ du}{\sqrt{E - V(u; a, c)}} \tag{2.5}$$

$$H(a, E, c) = \int_0^T \left(\frac{u_x^2}{2} - F(u) \right) dx = \frac{\sqrt{2}}{2} \oint_\Gamma \frac{E - V(u; a, c) - F(u)}{\sqrt{E - V(u; a, c)}} du$$

representing the mass, momentum, and Hamiltonian, respectively. As above, these integrals can be regularized at the square root branch points and hence represent C^1 functions of the traveling wave parameters. As seen in [6] and related papers, the gradients of the period, mass, and momentum play a large role in the stability of the periodic traveling wave solutions of (1.1). For notational simplicity then, we introduce the Poisson bracket

[2]The translation mode is simply inherited by the translation invariance of the governing PDE and can hence be modded out in our theory.

[3]In the case of the KdV or mKdV, the complete integrability of the PDE implies the existence of an infinite number of conservation laws. We will discuss this case more carefully in the following sections.

like notation

$$\{f,g\}_{x,y} = \det\left(\frac{\partial(f,g)}{\partial(x,y)}\right)$$

for two-by-two Jacobians, and $\{f,g,h\}_{x,y,z}$ for analogous three-by-three Jacobians.

It should be pointed out that in [6], it was shown when $E \neq 0$ gradients of the period can be interchanged for gradients of the conserved quantities via the relation

$$E\nabla_{a,E,c}T + a\nabla_{a,E,c}M + \frac{c}{2}\nabla_{a,E,c} + \nabla_{a,E,c}H = 0, \tag{2.6}$$

where $\nabla_{a,E,c} = \langle \partial_a, \partial_E, \partial_c \rangle$. Thus, although all results of this paper will be expressed in terms of Jacobians involving the quantities T, M, and P, it is possible to re-express them completely in terms of conserved quantities of the gKdV flow. Such an interpretation seems possibly more natural and desired from a physical point of view.

3 Evans Function Calculations

Now, suppose $u = u(\cdot; a, E, c)$ represents a particular periodic traveling wave solution of (1.1). As we are interested in studying the spectral stability of this solution to localized perturbations, we must study the linearized equation about u_0, namely

$$\partial_x \mathcal{L}[u]v = -v_t,$$

where $v \in L^2(\mathbb{R})$ and the operator $\mathcal{L}[u] = -\partial_x^2 - f'(u) + c$ is a self adjoint periodic Hill operator. Taking the Laplace transform in time yields the linearized spectral problem

$$\partial_x \mathcal{L}[u]v = \mu v, \tag{3.1}$$

where μ represents the Laplace frequency. We refer to the background solution u as being spectrally stable if the L^2 spectrum of the linear operator $\partial_x \mathcal{L}[u]$ is confined to the imaginary axis: notice the spectrum is symmetric about the real and imaginary axis.

The spectral analysis in $L^p(\mathbb{R})$, $1 \leq p < \infty$, of differential operators with periodic coefficients is known as Floquet theory. In this theory, one rewrites the linearized spectral problem (3.1) as a first order system of the form

$$\mathbf{Y}_x = \mathbf{H}(x,\mu)\mathbf{Y} \tag{3.2}$$

and lets $\Phi(x,\mu)$ be a matrix solution satisfying the initial condition $\Phi(0,\mu) = \mathbf{I}$ for all $\mu \in \mathbb{C}$, where $\mathbf{I}$ is the standard 3×3 identity matrix. The monodromy operator $\mathbb{M}(\mu)$, or the period map, is then defined to be

$$\mathbb{M}(\mu) := \Phi(T,\mu).$$

Notice that given any vector solution $\mathbf{Y}$ of the above first order system, the monodromy operator acts via the formula

$$\mathbb{M}(\mu)\mathbf{Y}(x,\mu) = \mathbf{Y}(x+T,\mu)$$

for all $x \in \mathbb{R}$ and $\mu \in \mathbb{C}$. In particular, it follows that the L^2 spectrum of the operator $\partial_x \mathcal{L}[u]$ is purely continuous and $\mu \in \text{spec}(\partial_x \mathcal{L}[u])$ if and only if

$$\det(\mathbb{M}(\mu) - e^{i\kappa}\mathbf{I}) = 0$$

for some $\kappa \in \mathbb{R}$. Such an equality is equivalent with the existence of a nontrivial solution v of (1.1) such that

$$v(x + mT) = e^{im\kappa} v(x) \tag{3.3}$$

for all $x \in \mathbb{R}$ and $m \in \mathbb{Z}$, which is equivalent with (3.1) admitting a nontrivial uniformly bounded solution.

Following Gardner (see [9] and [10]), we define the periodic Evans function for our problem to be

$$D(\mu, \kappa) = \det\left(\mathbb{M}(\mu) - e^{i\kappa}\mathbf{I}\right), \quad (\mu, \kappa) \in \mathbb{C} \times \mathbb{R}. \tag{3.4}$$

The complex constant $e^{i\kappa}$ is called the Floquet multiplier and the constant κ (which is not uniquely defined) is called the Floquet exponent[4]. In particular, by the above discussion we see $\mu \in \text{spec}(\partial_x \mathcal{L}[u])$ if and only if $D(\mu, \kappa) = 0$ for some $\kappa \in \mathbb{R}$. Moreover, since the coefficient matrix $\mathbf{H}(\mu, x)$ in (3.2) depends analytically on μ, it follows that D is analytic in the variables μ and κ and is real whenever μ is real.

From a computational viewpoint, however, it is more convenient to define the periodic Evans function by choosing a basis $\{\Phi_j(\cdot, \mu)\}_{j=1}^3$ of the kernel[5] of the linear operator $\partial_x \mathcal{L}[u] - \mu$ and noticing from (3.4) that

$$D(\mu, \kappa) = \left(\det(\Phi_j(0, \mu)\big|_{j=1}^3)\right)^{-1} \det\left(\Phi_j(T, \mu) - e^{i\kappa}\Phi_j(0, \mu)\big|_{j=1}^3\right). \tag{3.5}$$

This view is particularly convenient in our case; as a consequence of the integrability of the traveling wave ODE (2.1), one can easily verify that the set of functions $\{u_x, u_E, u_a\}$ is linearly independent and satisfy

$$\mathcal{L}[u]u_x = 0, \quad \mathcal{L}[u]u_E = 0, \quad \text{and} \quad \mathcal{L}[u]u_a = -1.$$

The first of these is a reflection of the translation invariance of (1.1). Thus, taking variations of the underlying periodic solution $u(\cdot; a, E, c)$ in the traveling wave parameters generate an explicit basis for the (formal) kernel of $\partial_x \mathcal{L}[u]$. In particular, one can easily verify that u_x and a linear combination of u_a and u_E constitute bonafide T-periodic elements of the kernel of $\partial_x \mathcal{L}[u]$, and hence $D(\mu, 0) = \mathcal{O}(|\mu|^2)$ for $|\mu| \ll 1$. Continuing, it follows again from (2.1) that $\partial_x \mathcal{L} u_c = -u_x$ and hence an appropriate linear combination of u_c, u_a, and u_E lies in the first T-periodic Jordan chain in the translation direction and hence $D(\mu, 0) = \mathcal{O}(|\mu|^3)$ for $|\mu| \ll 1$. Moreover, one can verify that if the Jacobian $\{T, M, P\}_{a,E,c}$ is non-zero, then

[4] By (3.3), the Floquet exponent encodes a class of admissible perturbations.

[5] Here, we are considering the formal operator $\partial_x \mathcal{L}[u]$ with out any reference to boundary conditions.

linear combinations of u_x, u_a, u_E, and u_c generate the entire generalized null space of $\partial_x \mathcal{L}[u]$ and hence the Evans function should indeed be of order $\mathcal{O}(|\mu|^3)$ near the origin.

Using perturbation theory one can then analyze the way these solutions bifurcate from the $(\mu, \kappa) = (0, 0)$ state by using (3.5) with $\Phi_1(x, 0)$, $\Phi_2(x, 0)$, and $\Phi_3(x, 0)$ corresponding to u_x, u_a, and u_E respectively. Moreover, it follows that the quantity

$$\frac{\partial}{\partial \mu} \Phi_1(x, \mu)\Big|_{\mu=0}$$

corresponds to a linear combination of u_c, u_a, and u_E, and that the first order variations in the u_a and u_E directions can be computed via variation of parameters. The details of this asymptotic calculation were carried out in Lemma 2 and Theorem 3 of [6], which we now recall as the main result of this section.

Theorem 1 ([6]). *Assume that $\{T, M, P\}_{a,E,c}$ is non-zero. Then in a neighborhood of $(\mu, \kappa) = (0, 0)$ the periodic Evans function admits the following asymptotic expansion:*

$$D(\mu, \kappa) = -\frac{\mu^3}{2}\{T, M, P\}_{a,E,c} + \frac{i\kappa\mu^2}{2}\left(\{T, P\}_{E,c} + 2\{M, P\}_{a,E}\right) + i\kappa^3 + \mathcal{O}(|\mu|^4 + \kappa^4).$$

Notice that from Theorem 1, spectral stability in a neighborhood of the origin is equivalent with the cubic polynomial

$$R(y) = -y^3 + \frac{y}{2}\left(\{T, P\}_{E,c} + 2\{M, P\}_{a,E}\right) - \frac{1}{2}\{T, M, P\}_{a,E,c} \tag{3.6}$$

having three real solutions, and hence modulational stability can be inferred from the discriminant of the polynomial R. Indeed, if this discriminant is positive then the polynomial R has three real roots and hence, using the Hamiltonian structure of the linearized operator $\partial_x \mathcal{L}[u]$, the spectrum in a neighborhood of the origin consists of a triple covering of the imaginary axis. If the discriminant is negative, however, then R must have a pair of complex roots $y_{1,2}$ with non-zero imaginary part and hence the spectrum in a neighborhood of the origin consists of the imaginary axis with multiplicity one, along with two curves which are tangent to lines through the origin with angle $\arg(iy_{1,2})$ to the imaginary axis. In particular, the underlying periodic traveling wave is modulationally unstable in the latter case. This is the main result of [6] concerning the stability of periodic traveling wave solutions of (1.1) to long wavelength perturbations.

It should be pointed out that by standard Abelian integral calculations, the Jacobians arising in Theorem 1 were shown in [7] to be explicitly computable in terms of the traveling wave parameters and moments of the underlying wave over a period in the case of a power law nonlinearity $f(u) = u^{p+1}$, $p \in \mathbb{N}$. For example, in the case of the KdV equation with $f(u) = u^2$, it was shown that the discriminant of R takes the form

$$\operatorname{disc}(R(\cdot)) = \frac{(\alpha_{3,0}T^3 + \alpha_{2,1}T^2M + \alpha_{1,2}TM^2 + \alpha_{0,3}M^3)^2}{\operatorname{disc}(E - V(\cdot; a, c))},$$

where the constants $\alpha_{j,3-j}$ represent nonlinear combinations of the traveling wave parameters a, E, and c. In particular, it follows that the underlying periodic traveling wave is modulationally stable provided the equation

$$E - V(y; a, c) = E + ay + \frac{c}{2}y^2 - \frac{1}{3}y^3 = 0$$

has three real solutions, which is equivalent with the constants (a, E, c) corresponding to a periodic orbit of the traveling wave equation (2.1). Since every periodic traveling wave solution of the KdV can be represented in terms of the Jacobbi elliptic function $\mathrm{cn}(x; \gamma)$, it follows that the cnoidal wave solutions of the KdV are always spectrally stable to long-wavelength perturbations as predicted by Whitham [28]. A similar representation holds in the case of the modified KdV equations with $f(u) = \pm u^3$ and hence a given periodic solution $u(\cdot; a, E, c)$ of the modified KdV is modulationally stable provided the polynomial equation

$$E + ay + \frac{c}{2}y^2 - \frac{1}{4}y^4 = 0$$

has four real solutions. While the same procedure can be implemented for other power law nonlinearities $f(u) = u^{p+1}$ with $p \in \mathbb{N}$, one must resort to numerical studies to compute the desired moments of the underlying periodic solution.

Finally, it should also be noticed that taking $\kappa = 0$ in the low frequency expansion of the Evans function provided in Theorem 1 suggests that the stability index

$$\mathrm{sgn}\ \ (D(\mu, 0)D(\Lambda, 0))$$

for real numbers $0 < \mu \ll 1 \ll \Lambda < \infty$ is determined[6] precisely by the Jacobian $\{T, M, P\}_{a,E,c}$. As a result, the underlying periodic traveling wave solution is spectrally unstable to co-periodic perturbations if $\{T, M, P\}_{a,E,c} < 0$, and (see [6]) is spectrally stable to such perturbations provided that $\{T, M, P\}_{a,E,c}$ and $T_E > 0$. In particular, the low-frequency expansion in Theorem 1 provides no information for the stability to co-periodic perturbations in the case where the Jacobian $\{T, M, P\}_{a,E,c}$ is zero, i.e. when the map

$$(a, E, c) \in \mathbb{R}^3 \to (T(a, E, c), M(a, E, c), P(a, E, c))$$

is not a local diffeomorphism. Thus, by (2.6), when $E \neq 0$ the nonvanishing of $\{T, M, P\}_{a,E,c}$ is equivalent with the statement that the conserved quantities of the PDE flow defined by (1.1) provide good local coordinates for nearby periodic traveling waves. This nondegeneracy condition has been seen in the stability analysis of periodic gKdV waves in several other contexts. For example, it appears in the nonlinear (orbital) stability analysis of such solutions [7] [13] to periodic perturbations, as well in the instability analysis to transverse perturbations in higher dimensional models [15]. In particular, using the elliptic integral methods [7] it has been seen that the quantity $\{T, M, P\}_{a,E,c}$ is generically nonzero for a wide variety of nonlinearities, including the most physically relevant examples of the KdV

[6] It was shown in [6] by elementary asymptotic ODE theory that $D(\Lambda, 0) < 0$ for $\Lambda > 0$ sufficiently large.

and modified KdV equations. As we will see in the next section, at the level of the Whitham system this condition is necessary for the resulting first order system of conservations to be of evolutionary type.

4 Slow Modulation Approximation

We now complement the rigorous results of the previous section with a formal Whitham theory calculation. In particular, we want to show that the linearized dispersion relation $R(\cdot)$ in (3.6) can be derived through a slow modulation (WKB) approximation, and hence that the formal homogenization procedures suggested by Whitham [27] [28] and Serre [26] correctly describe the stability of the periodic traveling wave solutions of (1.1) to long wavelength perturbations. To this end, recall from Section 2 that the gKdV admits a four parameter family $\mathcal{T}$ of periodic traveling wave solutions for some triple (a, E, c). We can thus form the quotient space $\mathcal{P} := \mathcal{T}/\mathcal{R}$ under the relation

$$(u \ \mathcal{R} \ v) \iff (\exists \xi \in \mathbb{R}; v = u(\cdot - \xi))$$

and we have the class functions

$$T = T(\dot{u}), \quad a = A(\dot{u}), \quad E = E(\dot{u}), \quad c = C(\dot{u}).$$

Similarly, since the conserved quantities are translation invariant it follows that M and P can be interpreted as class functions on $\mathcal{P}$ themselves.

Let (a_0, E_0, c_0) correspond to a particular non-constant periodic solution u_0 of (2.1) in $\mathcal{P}$, that is

$$\dot{u}_0(x) = \bar{v}(x; a_0, E_0, c_0)$$

for some $v \in \mathcal{P}$, where $a_0 = A(\dot{u}_0)$, $E_0 = E(\dot{u}_0)$, and $c_0 = C(\dot{u}_0)$. In particular, from the discussion in Section 2 it follows that since u_0 is nonconstant the projection

$$\begin{aligned} u &\mapsto \dot{u} \\ \mathcal{T} &\mapsto \mathcal{P} \end{aligned}$$

is locally a fibration (where the fibers are circles), and hence $\mathcal{P}$ is locally of dimension three. Now, consider equation (1.1) in the moving coordinate frame $x + c_0 t$:

$$u_t = u_{xxx} + f(u)_x - c_0 u_x. \tag{4.1}$$

A periodic solution of the corresponding traveling wave ODE is a stationary solution of (4.1), i.e. is a solution with wave speed $s = 0$ in this moving coordinate frame. Letting $\varepsilon > 0$, we now rescale (4.1) by the change of variables $(x, t) \mapsto (\varepsilon x, \varepsilon t)$ so that the rescaled equation takes the form

$$u_t = \varepsilon^2 u_{xxx} + f(u)_x - c_0 u_x, \tag{4.2}$$

where, with a slight abuse of notation, x and t now refer to the slow variables εx and εt.

Following Whitham then [27] [28], we consider a solution with a WKB type approximation of the form

$$u^\varepsilon(x,t) = u^0\left(x,t,\frac{\phi(x,t)}{\varepsilon}\right) + \varepsilon u^1\left(x,t,\frac{\phi(x,t)}{\varepsilon}\right) + \mathcal{O}(\varepsilon^2), \tag{4.3}$$

where $y \mapsto u^0(x,t,y)$ is an unknown 1-periodic function. It follows that the local period of oscillation of the function u^0 is $\varepsilon/\partial_x\phi$, where we assume the unknown phase *a priori* satisfies $\partial_x\phi \neq 0$. We now plug (4.3) into (4.2) and collect like powers of ε. The lowest power of ε present is ε^{-1} which leads to the equation

$$\phi_t\partial_y u^0 = (\phi_x\partial_y)^3 u^0 + (\phi_x\partial_y) f(u) - c_0 (\phi_x\partial_y) u^0.$$

Defining

$$s = -\frac{\phi_t}{\phi_x}, \quad \text{and} \quad \omega = \phi_x, \tag{4.4}$$

we find that the $\mathcal{O}(\varepsilon^{-1})$ equation is precisely the traveling wave ODE (2.1) in the variable ωy with wavespeed $c_0 - s$. In a similar way as above we may define class functions S and Ω on $\mathcal{P}$, respectively, associated to the quantities s and ω. Moreover, we may choose $u^0(x,t,\cdot)$ to be spatially periodic and satisfy the nonlinear oscillator equation

$$\frac{\left(u^0_y\right)^2}{2} = E + au^0 + \frac{c_0 - s}{2}\left(u^0\right)^2 - F(u^0), \tag{4.5}$$

where $a = a(x,t)$, $E = E(x,t)$, and $s = s(x,t)$ are independent of y. That is, we identify for each $(x,t) \in \mathbb{R}^2$ the projection of the function $y \mapsto u^0(x,t,y)$ into $\mathcal{P}$ as being a periodic traveling wave solution of the gKdV of the form

$$\dot{u}^0(x,t,y) = \bar{u}(y; A(\dot{u}^0), E(\dot{u}^0), c_0 - S(\dot{u}^0))$$

for some function $\bar{u} \in \mathcal{P}$, where we have used the notation from Section 2. In particular, notice that the parameters $(A(\dot{u}^0), E(\dot{u}^0), S(\dot{u}^0))$ defining $\dot{u}^0$ in $\mathcal{P}$ depend on the slow variables x and t introduced above.

Continuing, we find that the $\mathcal{O}(\varepsilon^0)$ equation reads

$$\partial_t u^0 = \partial_x f(u^0) - c_0\partial_x u^0 + \partial_x\left(\phi_x^2\partial_y^2 u^0\right) + \partial_y(\cdots).$$

Averaging over a single period in y and rescaling we find

$$\partial_t\left(M(\dot{u}^0)\Omega(\dot{u}^0)\right) - \partial_x G(\dot{u}^0) = 0, \tag{4.6}$$

where, using $T = T(\dot{u}^0)$ for convenience,

$$G(\dot{u}^0) = \frac{1}{T}\int_0^T \left(f(\dot{u}^0) - c_0\dot{u}^0 + \partial_y^2\dot{u}^0\right) dy.$$

Using (4.5) and the periodicity of u^0 in the variable y, it follows that

$$G(\dot{u}^0) = \frac{1}{T}\int_0^T \left(A(\dot{u}^0) - S(\dot{u}^0)\dot{u}^0\right) dy = \left(A - SM\Omega\right)(\dot{u}^0).$$

Moreover, the Schwarz identity $\phi_{xt} = \phi_{tx}$ provides us with the additional conservation law

$$\partial_t \Omega(\dot{u}^0) + \partial_x \left(S(\dot{u}^0)\Omega(\dot{u}^0)\right) = 0. \tag{4.7}$$

Now, as noted above, the manifold $\mathcal{P}$ has dimension three and hence equations (4.6) and (4.7) do not form a closed system for the unknown function $\dot{u}^0$. In [28] this problem was remedied in the case of the KdV equation by restricting to mean zero periodic waves, which forces the equation count to be correct. However, as noted in the introduction, this requirement can not be implemented in the general case considered here since the gKdV in general admits periodic waves of non-zero mean. In order to close the system then, we follow the approach of Serre [26] and augment the above equations with an additional conservation law arising from the group invariance under translation of the PDE flow generated by (1.1). The choice of this extra conservation law seems somewhat arbitrary where there is more than one to choose; for example, in the case of the modified KdV where complete integrability implies the existence of infinitely many conservation laws. We will return to this point in Section 5.

To identify a useful conserved quantity in our calculations, notice from Section 3 that the conserved quantity P must play a significant role in the modulational stability of the underlying periodic solution. With this as motivation, we notice from (4.1) that

$$\begin{aligned}\left(\frac{u^2}{2}\right)_t &= u\left(u_{xxx} + f(u)_x - c_0 u_x\right) \\ &= \left(uf(u) + uu_{xx} - F(u) - \frac{c_0}{2}u^2 - \frac{(u_y)^2}{2}\right)_x. \end{aligned} \tag{4.8}$$

Substituting (4.3) into (4.8) as before and collecting like powers of ε, we find that the $\mathcal{O}(\varepsilon^{-1})$ equation is the traveling wave ODE equation for u^0, as before, multiplied by the profile u^0, and averaging the $\mathcal{O}(\varepsilon^0)$ equation over a single period in y yields the conservation law

$$\partial_t \left(P(\dot{u}^0)\Omega(\dot{u}^0)\right) - \partial_x Q(\dot{u}^0) = 0, \tag{4.9}$$

where

$$Q(\dot{u}^0) = \frac{2}{T}\int_0^T \left(\dot{u}^0 f(\dot{u}^0) + \dot{u}^0\dot{u}^0_{yy} - F(\dot{u}^0) - \frac{1}{2}\left(\dot{u}^0_y\right)^2 - \frac{c_0}{2}(\dot{u}^0)^2\right) dy. \tag{4.10}$$

Again, using (4.5) one finds that

$$u^0 f(u^0) - F(u^0) = \frac{c_0 - s}{2}(u_0)^2 + \frac{\left(u^0_y\right)^2}{2} - E - u^0 u^0_{yy},$$

and hence (4.10) reduces to

$$\begin{aligned} Q(\dot{u}^0) &= -\frac{2}{T}\int_0^T \left(\frac{S(\dot{u}^0)}{2}\left(\dot{u}^0\right)^2 + E(\dot{u}^0)\right) dy \\ &= -\left(SP\Omega + 2E\right)(\dot{u}^0). \end{aligned} \tag{4.11}$$

The homogenized system (4.6), (4.7), and (4.9) is a closed system of three conservation laws in three unknowns, since $\dot{u}^0$ belongs to $\mathcal{P}$. In particular, (4.6)-(4.9) describe the mean behavior of the WKB approximation (4.3). This system is precisely the Whitham system we were seeking, and it can be written in closed form as

$$\partial_t\left(M\Omega, P\Omega, \Omega\right)(\dot{u}^0) - \partial_x\left(A - SM\Omega, -SP\Omega - 2E, -S\Omega\right)(\dot{u}^0) = 0. \tag{4.12}$$

We now wish to make a connection between the system (4.12) and the modulational stability of the original solution u_0. To this end, we make the assumption that the matrix

$$\frac{\partial(\dot{u}_0)}{\partial(a, E, s)}\Big|_{(a,E,s)=(a_0,E_0,0)} \tag{4.13}$$

is invertible, which implies that nearby periodic waves in $\mathcal{P}$ can be coordinatized by the traveling wave parameters (a, E, c) near (a_0, E_0, c_0). In particular, we can parameterize the Whitham system in terms of the variables (a, E, s) near $(a_0, E_0, 0)$ and hence

$$\partial_t\left(M\omega, P\omega, \omega\right)(a, E, s) - \partial_x\left(a - sM\omega, -sP\omega - 2E, -s\omega\right)(a, E, s) = 0, \tag{4.14}$$

where now we have dismissed the use of the class functions on $\mathcal{P}$ previously defined and work directly with the functions M, P, and $T = \omega^{-1}$ defined on $\mathbb{R}^3$ defined in section 2. Recall that Ω was the class function corresponding to the function $\omega : \mathbb{R}^3 \to \mathbb{R}$.

We now linearize (4.14) about the constant solution $(a, E, s) = (a_0, E_0, 0)$ corresponding to the original solution u_0 chosen above. To begin, notice from the integral representations (2.3)-(2.5) that

$$\partial_s\left\langle M, P, \omega\right\rangle = -\partial_c\left\langle M, P, \omega\right\rangle,$$

and hence assuming the quantity

$$\det\left(\frac{\partial(M\omega, P\omega, \omega)}{\partial(a, E, s)}\Big|_{(a,E,s)=(a_0,E_0,0)}\right) = \frac{1}{T^4}\{T, M, P\}_{a,E,c}(a_0, E_0, c_0)$$

is nonzero, the corresponding linearized system is of evolutionary type, i.e.the system (4.12) is of evolutionary type provided that the conserved quantities of the flow defined by (1.1) provide good local coordinates for the nearby periodic traveling waves. Recall from Section 2 that this property is generically true for several physically relevant nonlinearities, and is a major technical assumption in much of the current work on the stability of such solutions. For details, see again [6], [7], [13], and [15]. Moreover, as the linearization of the Whitham system (4.7) about the constant solution $(a_0, E_0, 0)$ is a constant coefficient system of PDE's, it can be solved by taking the Fourier-Laplace transform in space-time. In particular, it

follows the Cauchy problem for the linearized system is locally well posed provided that it is hyperbolic, which is equivalent with the characteristic equation

$$\det\left(\mu\frac{\partial(M\omega,P\omega,\omega)}{\partial(a,E,s)}-\frac{i\kappa}{T}\frac{\partial\left(a-sM\omega,-sP\omega-2E,-s\omega\right)}{\partial(a,E,s)}\right)(a_0,E_0,0)=0 \tag{4.15}$$

having three real roots in the variable $\frac{i\kappa}{\mu T}$, i.e. the matrix

$$\frac{\partial(M\omega,P\omega,\omega)}{\partial\left(a-sM\omega,-sP\omega-2E,-s\omega\right)}(a_0,E_0,0)$$

should be uniformly diagonalizable with real eigenvalues.

To make the connection between the above (formal) modulation theory calculation and the rigorous Evans function theory of the previous section, notice that (4.15) is equivalent to the vanishing of

$$\widetilde{R}(\mu,\kappa;a,E,s)=\det\left(\mu\frac{\partial(M\omega,P\omega,\omega)}{\partial(a,E,s)}-\frac{i\kappa}{T}\frac{\partial\left(a-sM\omega,-sP\omega-2E,-s\omega\right)}{\partial(a,E,s)}\right) \tag{4.16}$$

at the point $(a_0,E_0,0)$. Moreover, a straightforward computation shows that

$$\begin{aligned}\widetilde{R}(\mu,\kappa;a_0,E_0,0)&=\mu^3\det\left(\begin{pmatrix}M_a\omega & M_E\omega & -M_c\omega\\ P_a\omega & P_E\omega & -P_c\omega\\ -T_a\omega^2 & -T_E\omega^2 & T_c\omega^2\end{pmatrix}-\frac{i\kappa}{\mu T}\begin{pmatrix}1&0&0\\0&-2&0\\0&0&-\omega\end{pmatrix}\right)\\&=\frac{2\mu^3}{T^4}\left(-R\left(-\frac{i\kappa}{\mu}\right)+(2M_a-P_E-2T_c)\left(\frac{i\kappa}{\mu}\right)^2\right),\end{aligned}$$

where $R(\cdot)$ is the cubic polynomial given in (3.6) encoding the modulational stability of the periodic traveling wave solution $u(\cdot;a_0,E_0,c_0)$. At first sight, this result is disturbing as it suggests that the formal WKB/homogenization calculation does not accurately describe the modulational stability of such solutions. However, this apparent discrepancy can be easily explained using the integral representations of the period, mass, and momentum given in (2.3)-(2.5). Indeed, notice from this representation that

$$M_a=P_E=2T_c=-\frac{\sqrt{2}}{4}\oint_\Gamma\frac{u^2\,du}{\left(E-V(u;a,c)\right)^{3/2}}$$

and hence we have the relation $P_E+2T_c-2M_a=0$. In particular, it follows that

$$\widetilde{R}(\mu,\kappa;a_0,E_0,0)=-\frac{2\mu^3}{T^4}R\left(-\frac{i\kappa}{\mu}\right). \tag{4.17}$$

Notice that the root of the unexpected $\mu\mapsto-\mu$ conversion in (4.17) stems from the fact that in the Evans function calculation described in Section 3 we considered left moving waves (waves constant in the moving coordinate frame $x+ct$), while in homogenization

calculations of the above form it is customary to use right moving waves. Indeed, from our definition of s in (4.4) it follows that the quantity $\phi(x - st)$ is constant, i.e. ϕ corresponds to a *right* moving plane wave. Since changing right moving waves to left moving waves is equivalent to time reversal, which in turn arises as a $\mu \mapsto -\mu$ at the level of the linearized equations, this added negative sign seems necessary in the framework presented here. In particular, we have proven the following theorem.

Theorem 2. *Assume that $\{T, M, P\}_{a,E,c}$ is non-zero and that the matrix in* (4.13) *is non-singular. Then there exists a nonzero constant Γ_0 such that the periodic Evans function admits the following asymptotic expansion in a neighborhood of $(\mu, \kappa) = (0, 0)$:*

$$D(\mu, \kappa) = \Gamma_0 \widetilde{R}(-\mu, \kappa; a_0, E_0, 0) + \mathcal{O}(|\mu|^4 + \kappa^4).$$

That is, the linearized dispersion relation arising from the formal WKP approximation correctly describes the true linearized dispersion relation of the spectrum of the linearization about the underlying periodic traveling wave at the origin.

Corollary 1. *Under the assumptions of Theorem 2, a necessary condition for the spectral stability of the periodic traveling wave solution $u_0(\cdot) = u(\cdot; a_0, E_0, c_0)$ is that the spectrum of the three-by-three matrix*

$$\begin{aligned} A(a, E, s) &:= \left(\frac{\partial \left(a - sM\omega, -sP\omega - 2E, -s\omega \right)}{\partial(a, E, s)} \right)^{-1} \left(\frac{\partial(M\omega, P\omega, \omega)}{\partial(a, E, s)} \right) \\ &= \frac{\partial(M\omega, P\omega, \omega)}{\partial \left(a - sM\omega, -sP\omega - 2E, -s\omega \right)} \end{aligned}$$

be real at $(a_0, E_0, 0)$: equivalently, that the Whitham system (4.14) *be hyperbolic at $(a_0, E_0, 0)$.*

Proof. First, a straight forward calculation shows that the matrix

$$\frac{\partial \left(a + sM\omega, sP\omega - 2E, -s\omega \right)}{\partial(a, E, s)} \Big|_{(a,E,s)=(a_0,E_0,0)}$$

is invertible provided $\{T, M, P\}_{a,E,c}$ is non-zero at $(a, E, c) = (a_0, E_0, c_0)$ and the matrix in (4.13) is nonsingular at this special point. Moreover, if the matrix $A(a_0, E_0, 0)$ had a nonzero eigenvalue η, then it follows from Theorem 2 that there is a branch of spectrum bifurcating from the origin admitting an asymptotic expansion

$$\mu = \frac{i\eta\kappa}{T} + \mathcal{O}(\kappa^2)$$

for $|\kappa| \ll 1$. Thus, if η has non-zero imaginary part one immediately has a modulational instability of the underlying periodic traveling wave solution. □

It follows that we have established that a necessary condition for the spectral stability of a periodic traveling wave solution of (1.1) is that the Whitham system (4.6), (4.7),

and (4.9) be hyperbolic at the corresponding solution. Moreover, from the analysis in [6], if the matrix $A(a_0, E_0, 0)$ has only simple real eigenvalues, it follows that the underlying periodic wave is modulationally stable. Indeed, the Hamiltonian structure of the linearized operator $\partial_x \mathcal{L}[u]$ implies the spectrum is symmetric about the imaginary and real axes, and hence if the matrix $A(a_0, E_0, 0)$ has real distinct eigenvalues α_1, α_2, and α_3, then there are three branches $\mu_j(\kappa)$ which bifurcate from the origin with leading order expansions $\mu_j(\kappa) = \frac{i\alpha_j \kappa}{T} + \mathcal{O}(\kappa^2)$. If one did not have spectral stability near the origin, then two of the branches, say μ_1 and μ_2, would satisfy $\mu_1(\kappa) = -\overline{\mu_2(\kappa)}$. It follows by equating like powers of κ that $\alpha_1 = \alpha_2$, contradicting the simplicity of the spectrum of $A(a_0, E_0, 0)$. Thus, the strict hyperbolicity of the Whitham system is sufficient to conclude modulational stability of the underlying wave.

5 Discussion and open problems

In this paper, we have rigorously verified that the formal homogenization procedure introduced by Whitham to study the modulational stability of a periodic traveling wave profile of the gKdV equation does indeed describe the spectral stability near the origin. This in particular applies to the KdV case for which the WKB expansion is carried out in Chapter 14 of Whitham [28] by Lagrangian methods making use of integrability to carry out computations in terms of elliptic integrals. (The rigorous connection between the Lagrangian formulation and the type of WKB expansion carried out here is an important result of [28].)

Recall from Section 4 that there is at first sight a "missing equation" in the WKB equation for (KdV) and (gKdV). This is remedied in [28] by introducing a "potential" consisting of the anti-derivative of the solution, thus viewing the equation as a one higher order PDE for which the equation count is correct. Here, we follow instead the approach suggested by Serre [26] of augmenting the system by an additional conservation law associated with the integral of motion coming from Hamiltonian structure. This approach works not only in the integrable KdV and mKdV cases, but also for the general Hamiltonian (gKdV). The derivation of the Whitham equations seems also slightly simpler from this point of view. Indeed, using this approach of Serre together with identities developed by Bronski and Johnson [6], it seems remarkable that we are able to obtain the Whitham system (4.12) explicitly in terms of moments of the underlying periodic wave and the traveling wave parameters. Because of this, the verification that the hyperbolicity of the Whitham system correctly describes the spectral stability near the origin proved to be much simpler and straight forward than cases previously considered by Oh and Zumbrun [20] and Serre.

This issue of a missing equation comes up when there are more periodic solutions than would be generically expected [26], as arises when there is an integral of motion for the traveling wave profile ODE. Serre states on p. 262 of [26] that "a supplementary integral has a counterpart at the profile level", that is, such ODE integrals of motion often come from conserved quantities. The "supplementary integral" he refers to is exactly an additional conservation law, which he proposes to use as above to fill the same missing equation it causes. That is, this same basic approach, by the argument of Serre, should work in a wide

range of situations.

In situations such as the KdV, where there may be still further conservation laws, there is flexibility in deriving the Whitham system. However, all such derived systems must be equivalent. This would appear to correspond to existence for the Whitham system of additional conservation laws, or hyperbolic "entropies", another interesting possibility pointed out by Serre. We here just briefly investigate further the statement of Serre that supplementary PDE integrals should imply also supplementary traveling wave ODE integrals. We find justification for this statement in two very general situations. First, we consider a more general case in which by a supplementary integral, we mean an "entropy" in the conservation law sense, i.e., a function $\eta(u)$ of the unknown and no derivatives such that $\partial_t \eta(u) = \partial_x(...)$. In this case, we claim PDE integrals of motion give rise to integrals of motion of the corresponding traveling wave ODE. Indeed, note the quantity $\int \eta(u(x))dx$ is conserved in time, and that we have the additional traveling wave relation $-s\eta(u) + q(u) + C = L(u, u_x, ...)$, where C is a constant of integration and L is an expression of the same order as the traveling-wave ODE. However, using the existing equations we can eliminate the highest order derivative to get an algebraic relation between some lower derivatives, i.e., an integral of motion of the traveling-wave profile ODE. Secondly, in the more specialized Hamiltonian case, both properties can be inherited from an overarching Lagrangian structure, as in the multi-symplectic form system studied by many authors. See for example [2, 3]. This may tie in an interesting way to the variational approach of Witham, although we have not yet carried out such analysis.

Finally, we cite as open problems the rigorous verification of the Witham approximation at the level of behavior, either in the small ε limit as in [11], or in large-time behavior as in [23, 24, 25, 21]. Moreover, it is expected that a similar homogenization procedure can be used to justify the Whitham equations for the generalized Benjamin-Bona-Mahony equation, where the low frequency asymptotics were recently computed [14] in a similar way as to the gKdV case.

Acknowledgement. Many thanks to Jared Bronski for providing several useful insights into the connection of this work to that presented in [28] by Whitham. Also, we thank the reviewer for a careful and detailed check of the formulas and derivations presented in section four.

References

[1] E. Barthelemy and H. Michallet, *Experimental study of large interfacial solitary waves*, Journal of Fluid Mechanics, 336 (1998).

[2] T. Bridges and G. Derks, *The symplectic Evans matrix, and the instability of solitary waves and fronts,* Arch. Ration. Mech. Anal. 156 (2001), no. 1, 1–87.

[3] T. Bridges and G. Derks, *Constructing the symplectic Evans matrix using maximally analytic individual vectors,* Proc. Roy. Soc. Edinburgh A 133 505–526 (2003).

[4] T. L. Burch and A. R. Osborne, *Internal Solitons and the Andaman sea*, Science, 208 (1940), no. 4443.

[5] N. Bottman and B. Deconinck, *KdV cnoidal waves are linearly stable*, DCDS-A 25, (2009).

[6] J. C. Bronski and M. Johnson, *The modulational instability for a generalized Korteweg-de Vries equation*, Arch. Rat. Mech. Anal., DOI 10.1007/s00205-009-0270-5, (2009).

[7] J. C. Bronski, M. Johnson, and T. Kapitula, *An index theorm for the stability of periodic traveling waves of KdV type*, preprint.

[8] B. Deconinck and T. Kapitula, *On the orbital (in)stability of spatially periodic stationary solutions of generalized Korteweg-de Vries equations*, preprint.

[9] R. A. Gardner, *On the structure of the spectra of periodic travelling waves*, J. Math. Pures Appl. (9), 72 (1993), no. 5.

[10] R. A. Gardner, *Spectral analysis of long wavelength periodic waves and applications*, J. Reine Angew. Math. 491 (1997).

[11] O. Gues, G. Métivier, M. Williams, and K. Zumbrun, *Existence and stability of multidimensional shock fronts in the vanishing viscosity limit,* Arch. Ration. Mech. Anal. 175 (2005), no. 2, 151–244.

[12] M. Hărăguş and T. Kapitula, *On the spectra of periodic waves for infinite-dimensional Hamiltonian systems*, Physica D., 237(2008), no. 20.

[13] M. Johnson, *Nonlinear stability of periodic traveling wave solutions of the generalized Korteweg-de Vries equation*, Siam Journal on Mathematical Analysis, 41 no. 5, (2009).

[14] M. Johnson, *On the stability of periodic solutions of the generalized Benjamin-Bona-Mahony equation*, submitted.

[15] M. Johnson, *The transverse instability of periodic waves in Zakharov-Kuznetsov type equations*, Studies in Applied Math, DOI: 10.1111/j.1467-9590.2009.00473.x, (2010).

[16] B. C. Kalita and R. Das, *Modified Korteweg de Vries (mkdv) and Korteweg-de Vries (kdv) solitons in a warm plasma with negative ions and electrons' drift motion*, Journal of the Physical Socity of Japan, 71 (2002), no. 21.

[17] D. J. Korteweg and G. de Vries, *On the change of form of long waves advancing in a rectangular channel and on a new type of stationary waves*, Philosophical Magazine, 36(1895).

[18] A. A. Mamun, *Nonlinear propagation of ion-acoustic waves in a hot magnetized plasma with vortex-like electron distribution*, Physics of Plasmas, 5 (1998), no. 1.

[19] A. Mushtaq and H. A. Shah, *Study of non-maxwellian trapped electrons by using generalized (r,q) distribution function and their effects on the dynamics of ion acoustic solitary wave*, Physics of Plasmas, 13 (2006).

[20] M. Oh and K. Zumbrun, *Low-frequency stability analysis of periodic traveling-wave solutions of viscous conservation laws in several dimensions*, Journal for Analysis and its Applications, 25 (2006).

[21] M. Oh and K. Zumbrun, *Stability and asymptotic behavior of periodic traveling wave solutions of viscous conservation laws in several dimensions,* Arch. Rat. Mech. Anal., DOI 10.1007/s00205-009-0229-6, (2009).

[22] H. Segur and J. Hammack, *Soliton models of long internal waves*, Journal of Fluid Mechanics, 118 (1982).

[23] G. Schneider, *Nonlinear diffusive stability of spatially periodic solutions– abstract theorem and higher space dimensions*, Proceedings of the International Conference on Asymptotics in Nonlinear Diffusive Systems (Sendai, 1997), 159–167, Tohoku Math. Publ., 8, Tohoku Univ., Sendai, 1998.

[24] G. Schneider, *Diffusive stability of spatial periodic solutions of the Swift-Hohenberg equation,* (English. English summary) Comm. Math. Phys. 178 (1996), no. 3, 679–702.

[25] G. Schneider, *Nonlinear stability of Taylor vortices in infinite cylinders,* Arch. Rat. Mech. Anal. 144 (1998) no. 2, 121–200.

[26] D. Serre, *Spectral stability of periodic solutions of viscous conservation laws: large wavelength analysis*, Communications in Partial Differential Equations, 30 (2005), no. 1.

[27] G. B. Whitham, *Non-linear dispersive waves*, Proc. Roy. Soc. Ser. A, 283 (1965).

[28] G. B. Whitham, *Linear and Nonlinear Waves*, Pure and Applied Mathematics (New York), John Wiley & Sons Inc., New York, 1999. Reprint of the 1974 original, A Wiley-Interscience Publication.