Hints and Solutions: Determine the Polynomial

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First, let me restate the problem in a readable form: I have a polynomial $P(x)$ with positive integer coefficients. I claim that if you ask for the value of the polynomial at cleverly chosen points, then you can completely determine the polynomial from this information. How is this done?

1 Hint # 1

Choose as your first point something that will give you an upper bound on the size of each coefficient. Now what???

2 Harder Version: (Due to Kabe Moen and Jermey Martin at University of Kansas)

Come up with a way to do this (theoretically) by determining the value at only one point?

3 Solution

As your first point, choose $x = 1$. Since all the coefficients of $P(x)$ are positive and integers, we know $P(1)$ is greater than or equal to any of these coefficients. Now that you have $P(1)$, let 10^d be the least power of 10 greater than $P(1)$. For your second point, choose $x = 10^d$. Then you can read off the coefficients by reading $P(10^d)$ from right to left, taking d digits at a time. Then each grouping of d digits corresponds to a coefficient of x. To see how this works, it is probably best to see an example.

Example 1. *Suppose* $P(1) = 3,103$. *Then your next point should be* $x = 10,000$*, from which you get*

$$P(10,000) = 101,000,000,023,000.$$

Keeping in mind that no coefficient can be larger than 3103*, you can read the coefficients of each power of* x *by taking four numbers at a time, starting from left to right:*

$$101|0000|0002|3000$$

It follows that $P(x) = 101x^3 + 0x^2 + 2x + 3000$.

4 Solution to Harder Version

Thanks to Kabe Moen and Jermey Martin for providing the following interesting solution.

The easy way to do this is to determine the value of P at a given transcendental number, say $x = \pi$. Once you have the EXACT form of $P(\pi)$ (not just a truncated decimal approximation), then since π is transcendental you can just read off the (unique) coefficients of P.

For example, if $P(\pi) = 10\pi^3 + 2\pi^2 + 1$, then it must be the case that $P(x) = 10x^3 + 2x + 1$.

Admittedly this solution is not very practical, but it is very interesting from a theoretical standpoint!!